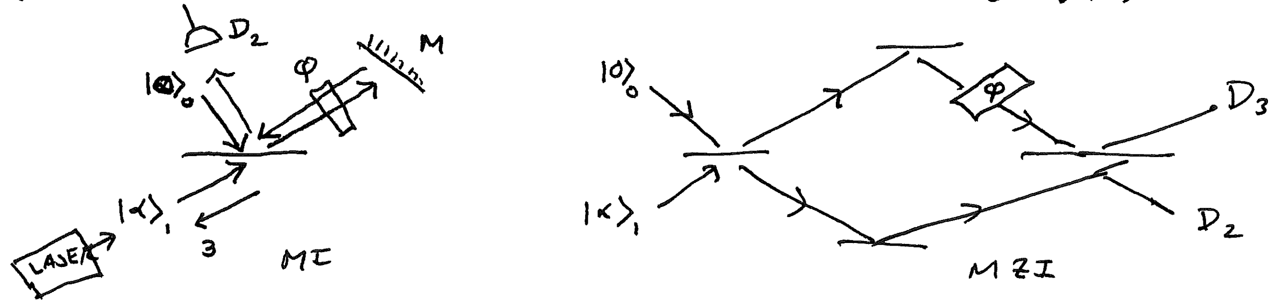
6.5 Coherent state Interferometer

A Michelson Interferometer (MI) is mathematically (SU(2)) equivalent to a Mach-Zehnder Interferometer (MZI)



Hence an MZI is called an "unfolded" MI which is easier to analyze. For LIGO the interferometer is balanced s.t. $\varphi = \frac{\pi}{2} + k\,l$ where l is the gravity wave displacement between two arms. Initially $l = 0$ and $P_3 = \cos^2(\pi/2) = 0$ and $P_2 = \sin^2(\pi/2) = 1$ so now 2 is the bright port and 3 the dark. For any $l < \lambda$ the balance is destroyed and $P_3 = \cos^2(\pi/2 + kl)$ and $P_2 = \sin^2(\pi/2 + kl)$ will give us information about l. So measuring φ is equivalent to measuring l our target.

Let us now send $|0\rangle_0 |\alpha\rangle_1 = |IN\rangle$ as our input state. Recall that an MZI behaves as a B.S. with φ-dependent t and r we recall:

$|OUT\rangle = |r\alpha\rangle_2 |t\alpha\rangle_3$ for a BS but for MZI

$r = e^{i\varphi/2} \cos\varphi/2$

$t = e^{i\varphi/2} \sin\varphi/2$

$$\Rightarrow \boxed{|OUT\rangle = |\alpha \cos\varphi/2\rangle_2 |\alpha \sin\varphi/2\rangle_3}$$

Ignoring an overall phase $ie^{i\varphi}$

Let us again define ~~d_2~~ $\hat{n}_2$ and ~~d_3~~ $\hat{n}_3$ the detector operators.

$$\langle \text{OUT} | \hat{d}_2 | \text{OUT} \rangle = I_2 = \bar{n} \cos^2 \varphi/2$$

$$\langle \text{OUT} | \hat{d}_3 | \text{OUT} \rangle = I_3 = \bar{n} \sin^2 \varphi/2$$

which exactly mimics the 1-photon case

Note $I_2 + I_3 = \bar{n}$ and energy is conserved

Hence if $\varphi = \pi/2$ all $|\alpha\rangle$ comes out ③ and none out ②.

A standard simple measurement scheme is to measure $\hat{d} \equiv \hat{n}_2 - \hat{n}_3$ analogous to "inversion" w and just the intensity difference between detectors

$$\langle \hat{d} \rangle = \bar{n} \cos^2 \varphi/2 - \bar{n} \sin^2 \varphi/2 = \boxed{\bar{n} \cos \varphi} = f_{\bar{n}}(\varphi)$$

We would now like to estimate the minimum detectable phase $\Delta\varphi$. Assume that $\langle \hat{d} \rangle = f_{\bar{n}}(\varphi)$ is strongly peaked at $\varphi = \pi/2$ and we can approximate $\frac{\Delta f}{\Delta \varphi} \approx \frac{df}{d\varphi}$ by differentials.

Or
$$\boxed{\Delta \varphi \approx \frac{\Delta f}{\left| \frac{df}{d\varphi} \right|}}$$

which is what we get from error propagation theory.

Let us interpret $\Delta f = \Delta d = \sqrt{\langle \hat{d}^2 \rangle - \langle \hat{d} \rangle^2}$ the variance of the quantum operator $\hat{d}$.

$$\langle \hat{d}^2 \rangle = \langle \text{OUT} | (\hat{n}_2 - \hat{n}_3)^2 | \text{OUT} \rangle \qquad [\hat{n}_2, \hat{n}_3] = 0$$

$$= \langle \text{OUT} | \hat{n}_2^2 - 2\hat{n}_2\hat{n}_3 + \hat{n}_3^2 | \text{OUT} \rangle$$

$$= \langle \text{OUT} | [\underbrace{\hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2 - 2\hat{n}_2\hat{n}_3 + \hat{a}_3^\dagger \hat{a}_3 \hat{a}_3^\dagger \hat{a}_3}_{\text{PUT IN NORMAL ORDER}}] | \text{OUT} \rangle$$

$$= \langle \text{OUT} | [\hat{a}_2^\dagger (\hat{a}_2^\dagger \hat{a}_2 + 1) \hat{a}_2 - 2\hat{n}_2\hat{n}_3 + \hat{a}_3^\dagger (\hat{a}_3^\dagger \hat{a}_3 + 1) \hat{a}_3] | \text{OUT} \rangle$$

$$= \langle \text{OUT} | [\hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + \underbrace{\hat{a}_2^\dagger \hat{a}_2 - 2\hat{n}_2\hat{n}_3 + \hat{a}_3^\dagger \hat{a}_3^\dagger \hat{a}_3 \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_3}_{\text{VACCUUM FLUCTUATIONS}}] | \text{OUT} \rangle$$

$$= \bar{n}^2 \cos^4\varphi/2 + \underbrace{\bar{n}\cos^2\varphi/2 - 2\bar{n}^2 \cos^2\varphi/2 \sin^2\varphi/2 + \bar{n}^2 \sin^4\varphi/2 + \bar{n}\sin^2\varphi/2}_{\text{VACCUM CONTRIBUTION}}$$

$$= (\bar{n}\cos^2\varphi/2 - \bar{n}\sin^2\varphi/2)^2 + \bar{n}$$

$$= \boxed{\langle \hat{d} \rangle^2 + \bar{n}}$$

vaccum term

$$\Rightarrow \quad \Delta d^2 = \langle \hat{d}^2 \rangle - \langle \hat{d} \rangle^2 = \boxed{\bar{n} = \text{vaccuum term}}$$

$$\Rightarrow \quad \Delta\varphi = \frac{\Delta d}{\left| \frac{\partial \langle d \rangle}{\partial \varphi} \right|} = \frac{\sqrt{\bar{n}}}{\bar{n}|\sin\varphi|} = \boxed{\frac{1}{\sqrt{\bar{n}}} \frac{1}{|\sin\varphi|}}$$

We have already seen $\varphi \approx \pi/2$ is our working point for a balanced MZI so

$$\boxed{\Delta\varphi = \frac{1}{\sqrt{\bar{n}}}}$$

Shotnoise Limit

$$\Rightarrow \quad \boxed{\Delta l = \frac{\lambda\!\!\!^{-}}{\sqrt{\bar{n}}}} \qquad \lambda\!\!\!^{-} = \frac{\lambda}{2\pi} \approx \frac{1.0\,\mu m}{2\pi}$$

minimum detectable path displacement. $\bar{n} = I =$ circulating power $\cong$ 1 kW! But Vaccum determines sensitivity!!!