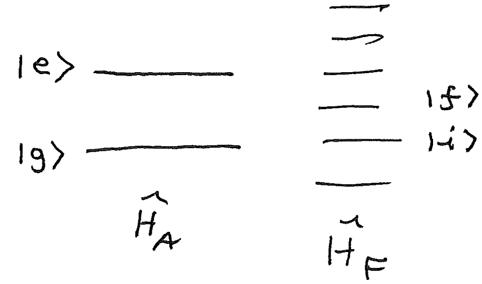
5.2 Quantum Coherence Theory

Let us model a detector as a two-level atom that becomes ionized.



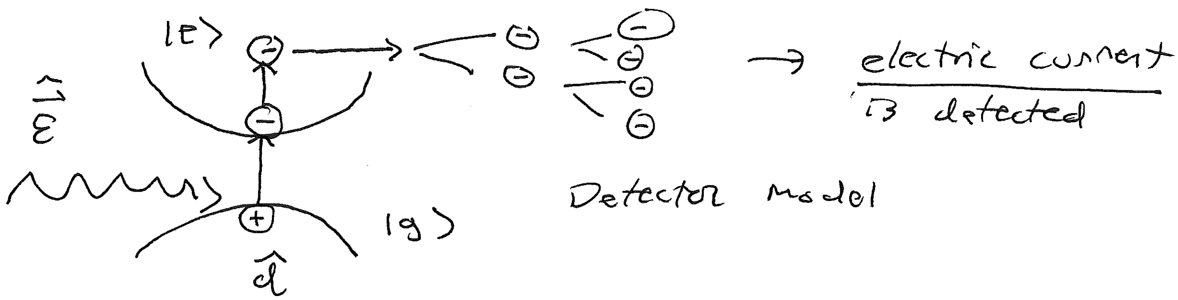
In this case

$$\hat{H}_A |g\rangle = E_g |g\rangle \quad \text{with } E_g < 0 \quad \text{bound}$$

$$\hat{H}_A |e\rangle = E_e |e\rangle \quad \text{with } E_e > 0 \quad \text{unbound}$$

For hydrogen $|g\rangle = |n l m\rangle$ and $|e\rangle = e^{i\vec{q}\cdot\vec{r}}$ a plane wave of momentum $\vec{p} = \hbar\vec{q}$ where $\vec{q}$ is wavenumber of free electron.

Ionized electron triggers an avalanch of electrons which are detected



Detector Model

With this techinique we can detect single photons — see Merzbacher photoelectric effect.

The interaction of field and detector atom is

$$\boxed{\hat{H}^{(I)} = -\hat{\vec{d}}\cdot\hat{\vec{\mathcal{E}}}(\vec{r},t)}$$

where for a plane wave

$$\hat{\vec{\mathcal{E}}}(\vec{r},t) = i\sum_{\vec{k}s} \mathcal{E}_0 \vec{e}_{\vec{k}s}\left[\hat{a}_{\vec{k}s}\cancel{e^{i\vec{k}\cdot\vec{r}}} - \hat{a}^{+}_{\vec{k}s}\cancel{e^{-i\vec{k}\cdot\vec{r}}}\right]$$

Dipole Approx

$$= \boxed{i\sum_{\vec{k}s} \mathcal{E}_0 \vec{e}_{\vec{k}s}\left[\hat{a}_{\vec{k}s} - \hat{a}^{+}\right]} \qquad \mathcal{E}_0^{(k)} = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$$

$$\Rightarrow \quad \hat{\vec{\mathcal{E}}}(\vec{r},t) \approx \hat{\vec{\mathcal{E}}}^{(+)} + \hat{\vec{\mathcal{E}}}^{(-)}$$

where

$$\hat{\vec{\mathcal{E}}}^{(\pm)} = i \sum_{\vec{k}s} \mathcal{E}_0(k)\, \vec{e}_{\vec{k}s} \begin{Bmatrix} \hat{a}_{\vec{k}s}(t) \\ \hat{a}^{+}_{\vec{k}s}(t) \end{Bmatrix}$$

where

$$\begin{Bmatrix} \hat{a}(t) \\ \hat{a}^{+}(t) \end{Bmatrix} = \begin{Bmatrix} \hat{a}\, e^{-i\omega t} & = \hat{a}\, e^{-i[+\omega]t} \\ \hat{a}^{+} e^{-i(-\omega)t} & = \hat{a}^{+} e^{-i[-\omega]t} \end{Bmatrix}$$

are the positive and negative frequency components. Only the (+) part $\propto \hat{a}$ can absorb a photon $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ hence we just drop the (−) part. This is equivalent to normal ordering.

Let $|I\rangle = |g\rangle|i\rangle$ be the initial atom-field state and $|F\rangle = |e\rangle|f\rangle$ the final. The transition matrix elt is

$$H^{(I)}_{FI} \equiv \langle F|\hat{H}^{(I)}|I\rangle = \langle e|\langle f|[-\hat{\vec{d}}\cdot\hat{\vec{\mathcal{E}}}]|g\rangle|i\rangle$$

$$= -\langle e|\hat{\vec{d}}|g\rangle \cdot \langle f|\hat{\vec{\mathcal{E}}}|i\rangle$$

$$\equiv \boxed{-\vec{d}_{eg}\cdot\vec{\mathcal{E}}_{fi}}$$

The transition probability is

$$P_{FI} \equiv |H^{(I)}_{FI}|^2$$

$$= |\vec{d}_{eg}|^2\, |\vec{\mathcal{E}}_{fi}|^2$$

$$\equiv P^{Atom}_{eg} \cdot P^{Field}_{fi}$$

Since we only measure state of <u>atom</u> we should trace over all final possible field states

$$P_{FI}^{Det.} = P_{eg}^{atom} \, Tr\{ P_{fi}^{Field} \}$$

$$= P_{eg}^{atom} \sum_f P_{fi}^{Field}$$

$$= P_{eg}^{atom} \sum_f |\langle f | \hat{\vec{\mathcal{E}}}^{(+)}(\vec{r},t) | i \rangle|^2$$

Let us call $P_{eg}^{atom} = d_{eg}^2 = d^2$ for short

also we have <u>dropped</u> the $\hat{\vec{\mathcal{E}}}^{(-)} \propto \hat{a}^+$ which does not contribute to an absorption event.

$$\sum_f |\langle f | \hat{\vec{\mathcal{E}}}^{(+)} | i \rangle|^2$$

$$\equiv \sum_f \langle f | \hat{\vec{\mathcal{E}}}^{(+)} | i \rangle^* \cdot \langle f | \hat{\vec{\mathcal{E}}}^{(+)} | i \rangle$$

$$= \underbrace{\sum_f \langle i | \hat{\vec{\mathcal{E}}}^{(-)} |f\rangle \cdot \langle f|}_{\mathbb{1}} \hat{\vec{\mathcal{E}}}^{(+)} | i \rangle$$

$$= \boxed{\langle i | \hat{\vec{\mathcal{E}}}^{(-)} \cdot \hat{\vec{\mathcal{E}}}^{(+)} | i \rangle}$$

$\hat{\mathcal{E}}^{(+)} \propto \hat{a}$

$\hat{\mathcal{E}}^{(-)} \propto \hat{a}^+$

$\Rightarrow \mathcal{E}^{\dagger(\pm)} = \hat{\mathcal{E}}^{(\mp)}$

This is $\sum_f P_{fi}^{Field}$ the probability the field makes a transition <u>some</u> where.

This calculation assumes $|i\rangle^{field}$ is pure.

If $|i\rangle^{\text{field}}$ is mixed then $|i\rangle_{\text{field}} \Rightarrow \hat{\rho}^{\text{field}}$

$$\equiv \boxed{\sum_i P_i\, |i\rangle\langle i|}$$

where P_i is the stat. mech. ensemble probability distribution for a mixed state

For example recall thermal state

$$P_i = \frac{\bar{n}^i}{(\bar{n}+1)^{i+1}} \quad \text{where} \quad \bar{n} = \frac{1}{e^{\hbar\omega/kT}-1} = \frac{1}{e^{x}-1} = \frac{1}{r^{-1}-1}$$

$$= \frac{r}{1-r} \quad \text{in old notes.}$$

Now

$$\sum_f P_{fi}^{\text{Field}} = \langle i|\,\hat{\vec{\mathcal{E}}}^{(-)}\cdot\hat{\vec{\mathcal{E}}}^{(+)}|i\rangle$$

$$\Rightarrow \boxed{\sum_i P_i \langle i|\,\hat{\vec{\mathcal{E}}}^{(-)}\cdot\hat{\vec{\mathcal{E}}}^{(+)}|i\rangle = \mathrm{Tr}\{\hat{\rho}_F(\hat{\vec{\mathcal{E}}}^{(-)}\cdot\hat{\vec{\mathcal{E}}}^{(+)})\}}$$

is the final (mixed) field states sum.

Using $x = (\vec{r}, t)$ in 4-vector notation, define a coherence term as

$$\boxed{G^{(1)}(x_i, x_j) \equiv \mathrm{Tr}\left[\hat{\rho}_{\text{Field}}\left(\hat{\mathcal{E}}^{(-)}_{(x_i)}\hat{\mathcal{E}}^{(+)}_{(x_j)}\right)\right]}$$

This is unnormalized intensity/coherence function as the trace is taken of the initial field $\hat{\rho}_{\text{Field}}$

$G^{(1)}$ is quantum version of time average $\langle\cdot\rangle_T$ classically.

Here we assume the field is polarized

s.t. $\hat{\vec{\mathcal{E}}}^{\pm}\cdot\hat{\vec{\mathcal{E}}}^{\pm} = \hat{\mathcal{E}}^{\pm}\hat{\mathcal{E}}^{\pm}\underbrace{\vec{e}\cdot\vec{e}}_{1}$

In Young's two slit we have two fields at detector

$$\hat{\mathcal{E}}^{(+)}(x) = \hat{\mathcal{E}}_1^{(+)} + \hat{\mathcal{E}}_2^{(+)} \equiv K_1 \hat{\mathcal{E}}^{(+)}(x_1) + K_2 \hat{\mathcal{E}}^{(+)}(x_2)$$

$$\hat{\mathcal{E}}^{(-)} \equiv \hat{\mathcal{E}}^{(+)\dagger} = \hat{\mathcal{E}}_1^{(-)} + \hat{\mathcal{E}}_2^{(-)} = K_1^* \hat{\mathcal{E}}^{(-)}(x_1) + K_2^* \hat{\mathcal{E}}^{(-)}(x_2)$$

The intensity $I(x)$ measured at $x = (\vec{r}, t)$ is

$$I(x) \equiv \mathrm{Tr}[\hat{\rho}(\hat{\mathcal{E}}^{(-)} \hat{\mathcal{E}}^{(+)})]$$

$$= \mathrm{Tr}[\hat{\rho}(\hat{\mathcal{E}}_1^{(-)} + \hat{\mathcal{E}}_2^{(-)})(\hat{\mathcal{E}}_1^{(+)} + \hat{\mathcal{E}}_2^{(+)})]$$

$$= \mathrm{Tr}[\hat{\rho}(\hat{\mathcal{E}}_1^{(-)}\hat{\mathcal{E}}_1^{(+)} + \hat{\mathcal{E}}_1^{(-)}\hat{\mathcal{E}}_2^{(+)} + \hat{\mathcal{E}}_2^{(-)}\hat{\mathcal{E}}_1^{(+)} + \hat{\mathcal{E}}_2^{(-)}\hat{\mathcal{E}}_1^{(+)}\}$$

$$= |K_1|^2 \underbrace{G^{(1)}(x_1, x_1)}_{G_{11}} + 2\,\mathrm{Re}\{K_1^* K_2 \underbrace{G^{(1)}(x_1, x_2)}_{G_{12}} + |K_2|^2 \underbrace{G^{(1)}(x_2, x_2)}_{G_{22}}$$

By comparison to classical theory Eq. 5.4

$|K_1|^2 G_{11} = I_1$ are single mode intensities

$|K_2|^2 G_{22} = I_2$ we generalize quantum coherence:

$$\boxed{g^{(1)}(x_1, x_2) \equiv \frac{G_{12}}{\sqrt{G_{11} G_{22}}}}$$

compare to classical $\gamma^{(1)}$

Let's define $g_{12} = |g^{(1)}(x_1, x_2)|$

$$g_{12} \begin{cases} \xrightarrow{0} \text{incoherent} \\ \xrightarrow{1/2} \text{partial} \\ \xrightarrow{1} \text{coherent} \end{cases}$$

$$G_{ii}^{(1)} \geq 0$$

$$G_{ij} = G_{ji}^*$$

$$G_{11} G_{22} \geq |G_{12}|^2$$ Triangle $\neq$

Examples $|n\rangle$

Assume plane wave $\boxed{\hat{\mathcal{E}}^{(+)} = i\,\mathcal{E}_0\,\hat{a}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}}$

$$\Rightarrow \quad \hat{\mathcal{E}}^{(-)} = -i\,\mathcal{E}_0\,\hat{a}^{+}\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\mathcal{E}_0 = \sqrt{\dfrac{\hbar\omega}{2\epsilon_0 V}}$ as before. $\Rightarrow \quad K_i = i\,e^{i(\vec{k}\cdot\vec{r}_i-\omega t_i)}$

Field in pure <u>number state</u> $|n\rangle \Rightarrow \hat{\rho}_n = |n\rangle\langle n|$

$$G^{(1)}(x,x) \equiv \mathrm{Tr}\left[\hat{\rho}_n\,\hat{\mathcal{E}}^{(-)}_{(x)}\,\hat{\mathcal{E}}^{(+)}_{(x)}\right]$$

$$= \sum_m \langle m|[\underbrace{|n\rangle}_{\delta_n{}^m}\underbrace{\langle n|\,\hat{\mathcal{E}}^{(-)}_{(x)}}\,\hat{\mathcal{E}}^{(+)}_{(x)}]|m\rangle$$

$$= \langle n|\,\hat{\mathcal{E}}^{(-)}_{(x)}\,\hat{\mathcal{E}}^{(+)}_{(x)}\,|n\rangle$$

$$= \langle n|\,\mathcal{E}_0^2\,\hat{a}^{+}\hat{a}\,|n\rangle = \boxed{n\,\mathcal{E}_0^2} = G^{(2)}(x_1 x_1) = G^{(2)}(x_2 x_2) = I_1 = I_2$$

This makes sense Intensity $= n \times$ single photon $I_0 = \mathcal{E}_0^2$

$$G_{12} = G^{(1)}(x_1 x_2) = \mathrm{Tr}\left[\hat{\rho}_n\,\hat{\mathcal{E}}^{(-)}_{(x_1)}\,\hat{\mathcal{E}}^{(+)}_{(x_2)}\right]$$

$$= \langle n|\,\hat{\mathcal{E}}^{(-)}_{(x_1)}\,\hat{\mathcal{E}}^{(+)}_{(x_2)}\,|n\rangle \qquad (\varphi_i \equiv \vec{k}\cdot\vec{r}_i - \omega t_i)$$

$$= \langle n|[-i\,\mathcal{E}_0 e^{-i\varphi_1}\,\hat{a}^{+}][+i\,\mathcal{E}_0\,e^{+i\varphi_2}\,\hat{a}\,]|n\rangle$$

$$= n\,\mathcal{E}_0^2\exp[i(\varphi_2-\varphi_1)]$$

$$= n\,\mathcal{E}_0^2\exp[i\,\Delta\varphi_{21}] \qquad \Delta\varphi_{21} = \vec{k}\cdot\underbrace{(\vec{r}_2-\vec{r}_1)}_{\Delta\vec{r}_{21}\,=\,\text{path diff}} - \omega\underbrace{(t_2-t_1)}_{\Delta t_{21}\,=\,\text{time diff}}$$

$$\Rightarrow \quad g^{(1)}(x_1 x_2) = \frac{G_{12}}{\sqrt{G_{11}\,G_{22}}} = \exp[i(\vec{k}\cdot(\Delta\vec{r}_{21}) - \omega\Delta t_{21})]$$

$$\Rightarrow \quad g_{12} \equiv |g^{(1)}(x_1 x_2)| = 1$$ <u>coherent!</u>

This is why interference with number states is possible Bill!

Example $|\alpha\rangle$

$$G_{12} = \bar{n}\,\mathcal{E}_0^2\, e^{i\Delta\varphi_{21}} \qquad |\alpha|^2 = \bar{n} = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle$$

$$G_{11} = G_{22} = \bar{n}\,\mathcal{E}_0^2$$

$$g^{(1)}(x_1 x_2) = \frac{\bar{n}\,\mathcal{E}_0^2}{\bar{n}\,\mathcal{E}_0^2}\, e^{i\Delta\varphi_{21}} = e^{i\Delta\varphi_{21}}$$

$$\Rightarrow \quad g_{12} = |g^{(1)}(x_1 x_2)|^2 = 1$$

coherent states are coherent!

Thermal state $\hat{\rho}_{Th} = \sum_n P_n^{Th} |n\rangle\langle n|$ Eq. 2.138

$$P_n^{Th} = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \quad \text{where} \quad \boxed{\bar{n} = \frac{1}{e^{\hbar\omega/kT}-1}}$$

is thermal average

$$G_{ii}(x_i x_i) = \mathrm{Tr}\left[\rho_n\, \hat{\mathcal{E}}^{(-)}_{(x)}\, \hat{\mathcal{E}}^{(+)}_{(x)}\right] \qquad x_i = x$$

$$= \sum_m \langle m| [\underbrace{\sum_n P_n |n\rangle\langle n|}_{\delta_{m}^{n}}\, \hat{\mathcal{E}}^{(-)}_{(x)}\, \hat{\mathcal{E}}^{(+)}_{(x)}]|m\rangle$$

$$= \sum_n P_n \langle n|\hat{\mathcal{E}}^{(-)}_{(x)}\, \hat{\mathcal{E}}^{(+)}_{(x)}|n\rangle$$

use $|n\rangle$ result

$$= \sum_n P_n\, n\, \mathcal{E}_0^2 = \mathcal{E}_0^2 \sum_n n P_n \equiv \boxed{\bar{n}\,\mathcal{E}_0^2}$$

Like a coherent state $\boxed{\bar{n}\,\mathcal{E}_0^2 = G_{11} = G_{22} = I_1 = I_2}$

$$G_{12} = \mathrm{Tr}\left[\rho_n\, \hat{\mathcal{E}}^{(-)}_{(x_1)}\, \hat{\mathcal{E}}^{(+)}_{(x_2)}\right]$$

$$= \sum_n P_n \langle n|\hat{\mathcal{E}}^{(-)}_{(x_1)}\, \hat{\mathcal{E}}^{(+)}_{(x_2)}|n\rangle$$

use n result

$$= \sum_n P_n\, \mathcal{E}_0^2\, n\, e^{i\Delta\varphi_{12}} = \boxed{\bar{n}\,\mathcal{E}_0^2\, e^{i\Delta\varphi_{12}}}$$

$$g^{(1)}(x_1 x_2) = G_{12}/\sqrt{G_{11}G_{22}} = e^{i\Delta\varphi_{12}}$$

$$\Rightarrow \quad g_{12} = |g^{(1)}(x_1 x_2)| = 1$$

Thermal Light is $\overset{(1)}{\text{coherent}}$ first order