4.6 Dressed States

The states $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$ are the bare states of $\hat{H}_{ATOM}$ and $\hat{H}_{Field}$ separately. The interaction $\hat{H}^{(I)}$ dresses the states. That is eigenstates of full hamiltonian $\hat{\mathcal{H}}$ are not bare states. Full J.C. Ham is:

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_I$$

$$= \boxed{\frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)}$$

IN Terms of bare states without RWA energy nonconserving terms allowed. Solution spanned by

$$\{\underbrace{|e\rangle|n\rangle,\ |g\rangle|n+1\rangle}_{\text{old bare}},\ \underbrace{|e\rangle|n+1\rangle,\ |g\rangle|n\rangle}_{\text{new bare}}\}$$

No other states can contribute. The Hamiltonian matrix elements are 4×4 Matrix. Restricting to RWA gives only $|i\rangle \equiv |e\rangle|n\rangle$; $|f\rangle \equiv |g\rangle|n+1\rangle$.

Want to find "new" eigenstates and "new" eigenvalues for "dressed" hamiltonian. Reduces to diagonalizing a 2×2 matrix [Nobel PRize 1997]

Compute $\langle i|\hat{H}|i\rangle$, etc.

Matrix Elts H_{ii}

Matrix Rep : $\hat{H} = \begin{bmatrix} H_{ii} & H_{if} \\ H_{fi} & H_{ff} \end{bmatrix}$ WLOG all Real

We note that $\hat{H}_I = \hat{H}_2^{\Delta=0}$ and use previous results

$$H_{ii} = \hbar(\omega_0/2 + n\omega)$$

$$H_{ff} = \hbar[-\omega_0/2 + (n+1)\omega]$$

$$H_{if} = H_{fi} = \hbar\lambda\sqrt{n+1}$$

The bare matrix representation is

$$\hat{H}_{\text{Bare.}} = \hbar\begin{bmatrix} \omega_0/2 + n\omega & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & -\omega_0/2 + (n+1)\omega \end{bmatrix} \equiv \begin{bmatrix} H_{ii} & H_{if} \\ H_{fi} & H_{ff} \end{bmatrix}$$

Find eigenvalues and eigen ~~values~~ vectors for 2x2 matrix. A bit tedious so we do using Mathematica attached: Recall $\Delta = \cancel{\omega-\omega_0}\ \omega_0 - \omega$

$$\Rightarrow \quad E_\pm = \frac{\hbar}{2}\left\{ (2n+1)\omega \pm \sqrt{\Delta^2 + 4(n+1)\lambda^2} \right\}$$

are the eigenvalues let

$$\Omega_n(\Delta) \equiv \sqrt{\Delta^2 + 4(n+1)\lambda^2} \quad \text{the detuned quantum Rabi}$$

$$\equiv \sqrt{\Delta^2 + \Omega_n^2} \equiv \text{"}R\text{" from before.}$$

$$\boxed{E_\pm \equiv \hbar(n+1/2)\omega \pm \hbar\,\Omega_n(\Delta)}$$

Diagonalized Hamiltonian is

$$\hat{H}_D = \begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix}$$

In[1]:= `hii = wo / 2 + n * w`

Out[1]= $n\,w + \frac{wo}{2}$

In[2]:= `hff = -wo / 2 + (n + 1) * w`

Out[2]= $(1+n)\,w - \frac{wo}{2}$

In[3]:= `hif = λ * Sqrt[n + 1]`

Out[3]= $\sqrt{1+n}\;\lambda$

Matrix Elts of $\hat{H}$ in $|i\rangle, |f\rangle$ basis

In[4]:= `evals = Eigenvalues[{{hii, hif}, {hif, hff}}]`

Out[4]= $\left\{\frac{1}{2}\left(w + 2\,n\,w - \sqrt{w^2 - 2\,w\,wo + wo^2 + 4\,\lambda^2 + 4\,n\,\lambda^2}\right),\ \frac{1}{2}\left(w + 2\,n\,w + \sqrt{w^2 - 2\,w\,wo + wo^2 + 4\,\lambda^2 + 4\,n\,\lambda^2}\right)\right\}$

In[5]:= `evals = FullSimplify[evals, Element[{w, wo, n, λ}, Reals]]`

$\hat{H}$ is diagonal in $|\pm\rangle$

Out[5]= $\left\{\frac{1}{2}\left(w + 2\,n\,w - \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}\right),\ \frac{1}{2}\left(w + 2\,n\,w + \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}\right)\right\}$

In[6]:= `evects = Eigenvectors[{{hii, hif}, {hif, hff}}]`

Out[6]= $\left\{\left\{-\frac{w - wo + \sqrt{w^2 - 2\,w\,wo + wo^2 + 4\,\lambda^2 + 4\,n\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\},\ \left\{-\frac{w - wo - \sqrt{w^2 - 2\,w\,wo + wo^2 + 4\,\lambda^2 + 4\,n\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\}\right\}$

In[7]:= `evects = FullSimplify[evects]`

Out[7]= $\left\{\left\{\frac{-w + wo - \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\},\ \left\{\frac{-w + wo + \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\}\right\}$

unnormalized evects

In[8]:= `InputForm[evects]`

Out[8]//InputForm=

```
{{(-w + wo - Sqrt[(w - wo)^2 + 4*(1 + n)*λ^2])/(2*Sqrt[1 + n]*λ),
  1}, {(-w + wo + Sqrt[(w - wo)^2 + 4*(1 + n)*λ^2])/
   (2*Sqrt[1 + n]*λ), 1}}
```

In[9]:= `plus = { (-w + wo - Sqrt[(w - wo) ^2 + 4 * (1 + n) * λ^2]) / (2 * Sqrt[1 + n] * λ), 1}`

Out[9]= $\left\{\frac{-w + wo - \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\}$ $\equiv |-\rangle$

In[10]:= `minus = { (-w + wo + Sqrt[(w - wo) ^2 + 4 * (1 + n) * λ^2]) / (2 * Sqrt[1 + n] * λ), 1}`

Out[10]= $\left\{\frac{-w + wo + \sqrt{(w-wo)^2 + 4\,(1+n)\,\lambda^2}}{2\sqrt{1+n}\;\lambda},\ 1\right\}$ $\equiv |+\rangle$

In[12]:= `FullSimplify[plus.minus]`

Out[12]= $0 = \langle + | - \rangle$

In[13]:= **normplus = FullSimplify[plus.plus]**

Out[13]= $$1 + \frac{\left(w - wo + \sqrt{(w - wo)^2 + 4\ (1 + n)\ \lambda^2}\right)^2}{4\ (1 + n)\ \lambda^2} \quad = \langle - | - \rangle$$

normminus = FullSimplify[minus.minus]

$$1 + \frac{\left(-w + wo + \sqrt{(w - wo)^2 + 4\ (1 + n)\ \lambda^2}\right)^2}{4\ (1 + n)\ \lambda^2} \quad = \langle + | + \rangle$$

Got my defs of "plus" and "minus" backwards.

Mathematica gives the un-normalized orthogonal evects

$|+\rangle \equiv A|i\rangle + B|f\rangle$; $|-\rangle \equiv C|i\rangle + D|f\rangle$ where

$$A = \frac{\Delta + R}{\Omega} \; ; \quad B = 1 \; ; \quad C = \frac{\Delta - R}{\Omega} \; ; \quad D = 1$$

and $R \equiv \sqrt{\Delta^2 + \Omega^2}$ with $\Delta \equiv \omega_0 - \omega$; $\Omega \equiv 2\lambda\sqrt{n+1}$

check orthogonality on mathematica $\langle +|-\rangle = \langle -|+\rangle \equiv 0$

To normalize compute

$$\langle +|+\rangle = |A|^2 + |B|^2 = \frac{(\overbrace{\Delta + R}^{\Delta_+})^2 + \Omega^2}{\Omega^2} \equiv \frac{\Delta_+^2 + \Omega^2}{\Omega^2}$$

$$\langle -|-\rangle = |C|^2 + |D|^2 = \frac{(\Delta - R)^2 + \Omega^2}{\Omega^2} \equiv \frac{\Delta_-^2 + \Omega^2}{\Omega^2}$$

Divide

$$A' \equiv \frac{A}{\sqrt{\langle +|+\rangle}} = \frac{\Delta_+}{\Omega}\sqrt{\frac{\Omega^2}{\Delta_+^2 + \Omega^2}} = \sqrt{\frac{\Delta_+^2}{\Delta_+^2 + \Omega^2}} = \boxed{\frac{\Delta_+}{\sqrt{\Delta_+^2 + \Omega^2}}}$$

$$B' = \frac{B}{\sqrt{\langle +|+\rangle}} = \boxed{\frac{\Omega}{\sqrt{\Delta_+^2 + \Omega^2}}}$$

$$C' = \frac{C}{\sqrt{\langle -|-\rangle}} = \frac{\Delta - R}{\cancel{\Omega}}\sqrt{\frac{\cancel{\Omega^2}}{\Delta_-^2 + \Omega^2}} = \boxed{\frac{\Delta_-}{\sqrt{\Delta_-^2 + \Omega^2}}}$$

$$D' = \frac{D}{\sqrt{\langle -|-\rangle}} = \boxed{\frac{\Omega}{\sqrt{\Delta_-^2 + \Omega^2}}}$$

we define

$|+\rangle' = A'|i\rangle + B'|f\rangle$ and $|-\rangle' \equiv C'|i\rangle + D'|f\rangle$

and check $\langle +|+\rangle' = A'^2 + B'^2 = 1$ ✓

$\langle -|-\rangle' = C'^2 + D'^2 = 1$ ✓

normalized.

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
AMPAD

To get form in book note $\Delta_\pm^2 + \Omega^2 = 2R\,[R \pm \Delta] = 2R\,\Delta_\pm$

$$\Rightarrow \quad A' = \sqrt{\frac{\Delta_+^2}{\Delta_+^2+\Omega^2}\ \frac{2R}{\Delta_+}\cdot\frac{\Delta_+}{2R}}$$

$$= \sqrt{\frac{\cancel{\Delta_+}}{\cancel{2R}\,\cancel{\Delta_+}}\ \cancel{2R}\cdot\frac{\Delta_+}{2R}} = \boxed{\pm\sqrt{\frac{\Delta_+}{2R}}}$$

$$B' = \sqrt{\frac{\Omega^2}{\Delta_+^2+\Omega^2}\ \frac{2R}{\Delta_-}\ \frac{\Delta_-}{2R}}$$

$$= \sqrt{\frac{\Omega^2}{\cancel{2R}\,\Delta_+}\ \frac{\cancel{2R}}{\Delta_-}\ \frac{\Delta_-}{2R}}$$

$$= \sqrt{\frac{\Omega^2}{R^2-\Delta^2}\ \frac{\Delta_-}{2R}} = \boxed{\pm\sqrt{\frac{\Delta_-}{2R}}}$$

since $R^2 - \Delta^2 = \Omega^2$

$$C' = \sqrt{\frac{\Delta_-^2}{\Delta_-^2+\Omega^2}\ \frac{2R}{\Delta_-}\ \frac{\Delta_-}{2R}} = \sqrt{\frac{\cancel{\Delta_-}\ \cancel{2R}}{\cancel{2R}\,\cancel{\Delta_-}}\ \frac{\Delta_-}{2R}} = \boxed{\pm\sqrt{\frac{\Delta_-}{2R}}}$$

where $\pm$ here denotes that $\Delta_\pm = \Delta \pm R$ can be plus or minus $\Rightarrow \sqrt{\Delta_\pm^2} = |\Delta_\pm|$

$$D' = \sqrt{\frac{\Omega^2}{\Delta_-^2+\Omega^2}\ \frac{2R}{\Delta_+}\ \frac{\Delta_+}{2R}}$$

$$= \sqrt{\frac{\Omega^2}{\cancel{2R}\,\Delta_-}\ \frac{\cancel{2R}}{\Delta_+}\ \frac{\Delta_+}{2R}} = \boxed{\pm\sqrt{\frac{\Delta_+}{2R}}}$$

WLOG we define Φ s.t.

$$A' = \cos[\Phi/2] \equiv +\sqrt{\frac{\Delta_+}{2R}}$$

$$B' = \sin[\Phi/2] \equiv +\sqrt{\frac{\Delta_-}{2R}}$$

$$C' = -\sin[\Phi/2] \equiv -\sqrt{\frac{\Delta_+}{2R}}$$

$$D' = +\cos[\Phi/2] \equiv +\sqrt{\frac{\Delta_-}{2R}}$$

With this sign convention the basis change from $|i\rangle, |f\rangle \rightarrow |\pm\rangle$ is a rotation

Energy Diagram

$$E_{\pm} = \hbar\omega(n+1/2) \pm \tfrac{1}{2}\hbar R = \hbar\left[\omega(n+1/2) \pm \sqrt{\Delta^2+\Omega_n^2}\,\right]$$

where $R = \sqrt{\Delta^2+\Omega_n^2}$ is Rabi $\Omega_n(\Delta)$ in book.

Since $\Omega_n = 2\lambda\sqrt{n+1}$ we say that $|i\rangle = |e\rangle|n\rangle$ and $|f\rangle = |g\rangle|n+1\rangle$ are the uncoupled bare states and $|\pm\rangle$ are the dressed states. Calculation is identical to Stark shift.

Recall
$$E_i = E_e + E_n = +\frac{\hbar\omega_0}{2} + \hbar\omega n$$
$$E_f = E_g + E_{n+1} = -\frac{\hbar\omega_0}{2} + \hbar\omega(n+1)$$

Hence "bare" splitting is

$\cancel{\Delta E_{fi} = E_f - E_i = -\hbar\omega_0 + \hbar\omega =}$

$\cancel{\Delta E_{fi} = E_f - E_i = \hbar}$

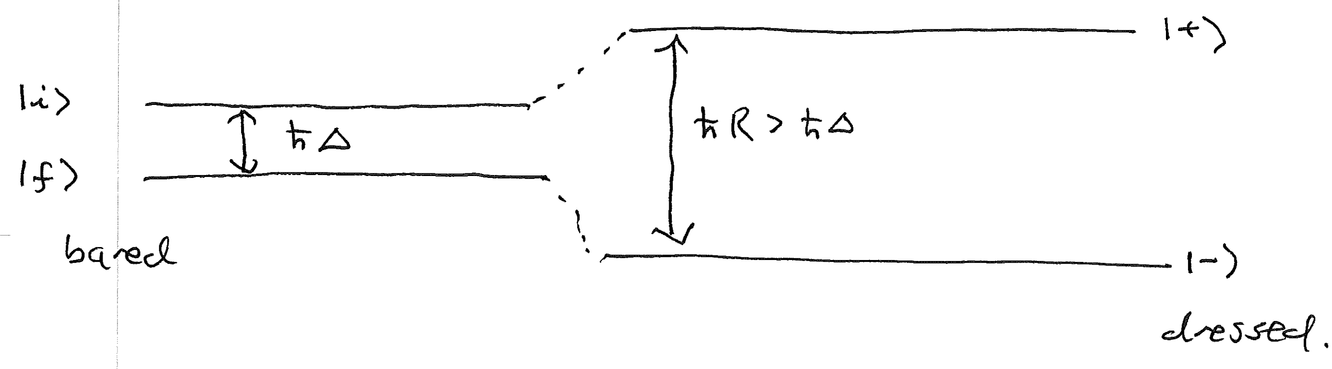
$$\Delta E_{if} = E_i - E_f = \hbar\omega_0 - \hbar\omega = \boxed{\hbar\Delta}$$

Hence bare states are degenerate if $\Delta = 0$. Interaction causes AC-Stark splitting

$$\Delta E_{\pm} \equiv E_+ - E_- = \hbar R = \hbar\sqrt{\Delta^2+\Omega_n^2} = \hbar\sqrt{\Delta^2+4\lambda^2(n+1)} > \hbar\Delta$$

so dressed levels are farther apart and are split even when $\Delta = 0$ and $n = 0$

$$\Delta E_{\pm}\Big|_{\Delta=0} = 2\hbar\lambda\sqrt{n+1}\,\Big|_{n=0} = \boxed{2\hbar\lambda}$$ vacuum Rabi splitting



In $\Delta = 0$ limit

$$A' = B' = C' = D' = \pm 1/\sqrt{2}$$

$$\boxed{\begin{aligned} |+\rangle_n &= [\,|e\rangle|n\rangle + |g\rangle|n+1\rangle\,]/\sqrt{2} \\ |-\rangle_n &= [\,-|e\rangle|n\rangle + |g\rangle|n+1\rangle\,]/\sqrt{2} \end{aligned}}$$

Dressed states
$\Delta = 0$

Let's assume field is initially in ~~vacuum $|n\rangle = |0\rangle$~~ numberstate $|n\rangle$.
and atom excited so $|i\rangle = |e\rangle|n\rangle = |e\rangle|n\rangle$.
Let the detuning be zero $\Delta = 0$, using above equations we can invert dressed for bare

$$|+\rangle + |-\rangle = \frac{2|g\rangle|n+1\rangle}{\sqrt{2}} \Rightarrow \boxed{|g\rangle|n+1\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}}$$

and
$$\boxed{|e\rangle|n\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}}$$

The system at $t=0$ is in the latter

$$|\psi(t=0)\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Since $|\pm\rangle$ are stationary states of the full hamiltonian
$\hat{H}|\pm\rangle = E_\pm|\pm\rangle$ we can now solve TDSE

$$i\hbar|\pm\rangle^{\bullet} = \hat{H}|\pm\rangle = E_\pm|\pm\rangle \Rightarrow \boxed{|\pm\rangle_{t>0} = e^{-i\frac{1}{\hbar}E_\pm t}|\pm\rangle_0}$$

Hence the bare state evolution is trivial

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{1}{\hbar}E_+ t}|+\rangle - e^{-i\frac{1}{\hbar}E_- t}|-\rangle\right]}$$

where converting back to $|i\rangle$, $|f\rangle$
gives previous solutions Eq. 4.112