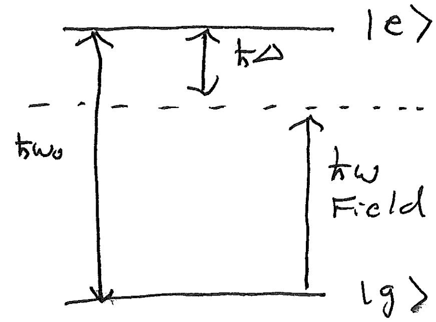
4.4 Rabid Rabi Model

TDPT assumes $|C_f| << |C_i|$ but for a strong field this is not true right from a laser example. For a two-level atom in <u>semiclassical</u> theory it can be solved exactly! (See Merzbacher)

Note $\vec{\mathcal{E}}_0$ <u>classical</u>

$\vec{\mathcal{E}}_0 \cos\omega t$



$$\hat{H}^I(t) \equiv \hat{\vec{V}}_0 \cos\omega t$$

$$\hat{\vec{V}}_0 \equiv -\hat{\vec{d}} \cdot \vec{\mathcal{E}}_0 \quad \leftarrow \text{no hat}$$

$|excite\rangle = |e\rangle$

$|ground\rangle = |g\rangle$

$$\hat{H} = \hat{H}_0 + \hat{H}^I(t) = \hat{H}_0 + \hat{\vec{V}}_0 \cos\omega t$$

$$\hat{H}_0 |e\rangle = E_e |e\rangle = \hbar\omega_e |e\rangle$$
$$\hat{H}_0 |g\rangle = E_g |g\rangle = \hbar\omega_g |g\rangle$$

$\Rightarrow$ $\boxed{\omega_0 \equiv \omega_e - \omega_g}$ Resonance

$\boxed{\Delta \equiv \omega_0 - \omega}$ Detuning

$\Delta = 0$ on resonance atom-field

$\Delta \neq 0$ off resonance

Let $|t\rangle$ solve TDSE

$$i\hbar|\dot{t}\rangle = \hat{H}|t\rangle = \hat{H}_0|t\rangle + \hat{H}^I(t)$$
$$= \hat{H}_0|t\rangle + \hat{V}_0 \cos\omega t\, |t\rangle$$

where

$$\boxed{|t\rangle \equiv C_g(t)\, e^{-i\omega_g t} |g\rangle + C_e(t)\, e^{-i\omega_e t} |e\rangle}$$

That is Eq. 4.20 only has two terms.

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\left\{ C_g e^{-i\omega_g t}|g\rangle + C_e e^{-i\omega_e t}|e\rangle \right\} = LHS$$

$$= i\hbar \left\{ [\dot{C}_g - i\omega_g C_g] e^{-i\omega_g t}|g\rangle + [\dot{C}_e - i\omega_e C_e] e^{-i\omega_e t}|e\rangle \right\}$$

$$\Rightarrow \quad LHS = [i\hbar \dot{C}_g + E_g C_g] e^{-i\omega_g t}|g\rangle + [i\hbar \dot{C}_e + E_e C_e] e^{-i\omega_e t}|e\rangle$$

$$RHS = \hat{H}|t\rangle = \hat{H}[C_g e^{-i\omega_g t}|g\rangle + C_e e^{-i\omega_e t}|e\rangle]$$

$$= E_g C_g e^{-i\omega_g t}|g\rangle + E_e C_e e^{-i\omega_g t}|e\rangle + \hat{H}^I|t\rangle$$

LHS = RHS

$$\Rightarrow \quad i\hbar [\dot{C}_g e^{-i\omega_g t}|g\rangle + \dot{C}_e e^{-i\omega_e t}|e\rangle]$$

$$= \hat{H}^I|t\rangle = \hat{V}_0 \cos\omega t \, [C_g e^{-i\omega_g t}|g\rangle + C_e e^{-i\omega_e t}|e\rangle]$$

$$\Rightarrow \quad \left\{ \begin{matrix} \langle g| \rightarrow \\ \langle e| \rightarrow \end{matrix} \right\} \quad i\hbar \dot{C}_g e^{-i\omega_g t} = \cancel{V_{gg}}^{\rightarrow 0} \cos\omega t \, C_g e^{-i\omega_g t} + V_{ge} \cos\omega t \, C_e e^{-i\omega_e t}$$

$$\& \quad i\hbar \dot{C}_e e^{-i\omega_e t} = V_{eg} \cos\omega t \, C_g e^{-i\omega_g t} + \cancel{V_{ee}}^{\rightarrow 0} C_e e^{-i\omega_e t}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \dot{C}_g &= -\frac{i}{\hbar} C_e V_{ge} \cos\omega t \, e^{-i\omega_0 t} \\ \dot{C}_e &= -\frac{i}{\hbar} C_e V_{ge} \cos\omega t \, e^{+i\omega_0 t} \end{aligned}}$$

$$\omega_0 \equiv \omega_e - \omega_g$$

$$V_{ge} = V_{eg} = \langle e|\hat{V}|g\rangle = \langle g|\hat{V}|e\rangle \equiv \nu$$

we assume WLOG $V_{eg} = V_{ge} \in \mathbb{R}$eal

Recall $V_{ee} = V_{gg} = 0$ as $\langle \vec{d} \rangle$ is zero.

$$(\cos\omega t)\, e^{\pm i\omega_0 t} = \tfrac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right] e^{\pm i\omega_0 t}$$

$$= \tfrac{1}{2}\left[e^{i(\omega\pm\omega_0)t} + e^{-i(\omega\mp\omega_0)t}\right] \quad \left\{\begin{matrix}+\\-\end{matrix}\right\}$$

$$= \left\{\begin{matrix}\tfrac{1}{2}\, e^{+\Delta t}\\ \tfrac{1}{2}\, e^{-\Delta t}\end{matrix}\right. \qquad \text{where} \quad \boxed{\Delta \equiv \omega_0 - \omega}$$

DROP TERMS WITH $\omega+\omega_0$ RWA.

$$\Rightarrow \boxed{\begin{aligned}\dot{c}_g &= -iU e^{-i\Delta t} c_e\\ \dot{c}_e &= -iU e^{+i\Delta t} c_g\end{aligned}} \qquad U \equiv \frac{\nu}{2\hbar}$$

Two coupled Diffy Q

$$\Rightarrow \quad c_g = \frac{i}{U} e^{-i\Delta t} \dot{c}_e$$

$$\Rightarrow \quad \dot{c}_g = \frac{i}{U}\left[-i\Delta \dot{c}_e + \ddot{c}_e\right] e^{-i\Delta t}$$

$$\Rightarrow \quad -iU \cancel{e^{-i\Delta t}} c_e = \left[\frac{\Delta}{U}\dot{c}_e + \frac{i}{U}\ddot{c}_e\right]\cancel{e^{-i\Delta t}}$$

$$\Rightarrow \quad U^2 c_e = i\Delta \dot{c}_e - \ddot{c}_e$$

$$\Rightarrow \quad \boxed{\ddot{c}_e - i\Delta\dot{c}_e + U^2 c_e = 0}$$

Note if $U^2 \ll 1$
Weak Field Limit
We get previous result

Let $e^{i\lambda t}$ be a guess

$$\Rightarrow \quad -\lambda^2 + \lambda\Delta + U^2 = 0 \quad \Rightarrow \quad \lambda^2 - \Delta\lambda - U^2 = 0$$

$$\lambda = \tfrac{1}{2}\Big[\Delta \pm \underbrace{\sqrt{\Delta^2 - 4U^2}}_{R}\Big] \equiv \lambda_\pm = \tfrac{1}{2}\left[\Delta \pm R\right]$$

Hence two independent solutions are.

$$C_e(t) = A_+ e^{i\lambda^+ t} + A_- e^{+i\lambda^- t}$$

B.C. $\quad C_e(0) = 0 \qquad C_g(0) = 1$

$$\Rightarrow \quad 0 = A_+ + A_- \Rightarrow A_+ = -A_- \equiv -A \in \text{Real WLOG}$$

$$\boxed{C_e(t) = A\left[e^{i\lambda^+ t} - e^{i\lambda^- t}\right]}$$

$$\Rightarrow \quad \dot{C}_e(t) = A\left[i\lambda^+ e^{i\lambda^+ t} - i\lambda^- e^{i\lambda^- t}\right] \equiv -iu\, e^{i\Delta t} \underbrace{C_g(t)}_{\to 1} \Big|_{t=0}$$

$$\Rightarrow \quad A\left[i\lambda^+ - i\lambda^-\right] \equiv -iu$$

$$\Rightarrow \quad A = \frac{u}{\lambda^- - \lambda^+} = \frac{u}{\frac{1}{2}\left[\left[\Delta + \sqrt{R}\right] - \left[\Delta - R\right]\right]} = \frac{U}{R} = \frac{U}{\Omega_R}$$

$$\Rightarrow \quad \boxed{A = \frac{\nu}{2\hbar\sqrt{\Delta^2 - 4u^2}} = \frac{\nu}{2\hbar}\left(\Delta^2 - \frac{\nu^2}{\hbar^2}\right)^{-1/2}}$$

$$C_e(t) = A\left[e^{i(\Delta+R)/2\, t} - e^{i(\Delta-R)/2\, t}\right]$$

$$\boxed{= 2iA\, e^{i\Delta t/2} \sin\left[\Omega_R t/2\right]}$$

$R = \Omega_R$

$\equiv \sqrt{\Delta^2 + 4u^2}$

$\equiv$ Rabi Frequ

$$\Rightarrow \quad \boxed{P_e(t) = |C_e|^2 = 4A^2 \sin^2\left[\Omega_R t/2\right]} \quad \boxed{= \frac{4U^2}{\Omega_R^2} \sin^2\left[\frac{\Omega_R t}{2}\right]}$$

$\downarrow U \ll 1$

Probability oscillates at freq.

$$\Omega_R = \sqrt{(\omega_0 - \omega)^2 + \frac{\nu^2}{4\hbar^2}}$$

$$\underbrace{\cong 4 \sin^2\left[\frac{\Delta t}{2}\right]}_{\text{Weak Field Result}}$$

$$c_g = \frac{i}{u} e^{-i\Delta t}\, \dot{c}_e$$

$$= \frac{i}{u} e^{-i\Delta t}\left[2iA\left\{\frac{i\Delta}{2}\sin\frac{\Omega t}{2} + \frac{\Omega}{2}\cos\frac{\Omega t}{2}\right\} e^{i\Delta t/2}\right]$$

$$= \frac{2ii}{u} A\, e^{-i\Delta t/2}\left\{\frac{i\Delta}{2}\sin\frac{\Omega t}{2} + \frac{\Omega}{2}\cos\frac{\Omega t}{2}\right\}$$

$$= \boxed{\frac{1}{\Omega_R} e^{-i\Delta t}\left\{i\Delta \sin\frac{\Omega t}{2} + \Omega\cos\frac{\Omega t}{2}\right\}}$$

$$\Rightarrow \boxed{P_g = |c_g|^2 = \frac{1}{\Omega_R^2}\left[\Delta^2\sin^2\frac{\Omega t}{2} + \Omega^2\cos^2\frac{\Omega t}{2}\right]}$$

$$P_e + P_g = \left[\left(\frac{\overbrace{4U^2+\Delta^2}^{\Omega_R^2}}{4\Omega_R^2}\right)\sin^2\frac{\Omega t}{2} + \cancel{\tfrac{1}{4}}\cos^2\frac{\Omega t}{2}\right]$$

$$= [1] \checkmark$$ off by $\left(\frac{1}{2}\right)^2$

IF $\Delta = 0 \Rightarrow \Omega = 4U^2 \Rightarrow P_e = \sin^2[Ut]\Big|_{\Delta=0}$

$P_g = \cos^2[Ut]\Big|_{\Delta=0}$

Hence for $t = T_\pi = \frac{\pi}{2U}$

$$= \frac{\pi}{2\frac{\nu}{2\hbar}} = \frac{\pi}{4\nu} = \boxed{\frac{\pi}{4\nu}}$$

A pulse with integrated E field gives all population in $|e\rangle$.

$$W(t) \equiv P_e(t) - P_g(t)$$ Atomic Inversion

$$= \boxed{\left(\frac{4\upsilon^2 - \Delta^2}{\Omega}\right) \sin^2\left(\frac{\Omega t}{2}\right) - \cos^2\left(\frac{\Omega t}{2}\right)}$$

$W = 1$ inverted

$W = -1$ ground

Note $$\lim_{t \to \infty} W(t) = \frac{1}{2}\left(\frac{4\upsilon^2 - \Delta^2}{4\upsilon^2 + \Delta^2}\right) - \frac{1}{2}$$

$$= \begin{cases} 0 \; ; & \Delta = 0 \\ -1 \; ; & \Delta = \infty \end{cases}$$

So at most 50/50 never inverted on average

No laser!

$$W_{\Delta=0}(t) = \frac{4\cancel{\upsilon^2}}{4\cancel{\upsilon^2}} \sin^2 - \cos^2 = \boxed{-\cos[2\upsilon t]}$$

$$t = T_\pi = \frac{\pi}{2\upsilon} \Rightarrow \boxed{W_{\Delta=0} = -\cos\pi = +1}$$

If $t = \frac{T_\pi}{2} = \frac{\pi}{4\upsilon}$ $\quad W_{\Delta=0} = -\cos\pi/2 = 0$

Qubit

$$\Rightarrow C_e(T_\pi/2) = \frac{i}{\sqrt{2}} \; ; \; C_g = \frac{1}{\sqrt{2}} \Rightarrow \boxed{|\Psi(T_\pi/2)\rangle = \frac{|g\rangle + i|e\rangle}{\sqrt{2}}}$$