# 4.3 Quantized Atom + Quantized Field

Note in semiclassical $P_{fi} \propto |H^I_{fi}|^2 = |H^I_{if}|^2 \propto P_{if}$

Engineers call the reciprocity. If $i > f$

this is probability of stimulated

emission $|H^I_{if}|^2 \propto |\vec{\mathcal{E}}_0 \cdot \vec{d}_{if}|^2 = |\mathcal{E}_0 \cdot d_{fi}|$

Hence down $\downarrow^{i}_{f}$ and $\uparrow^{f}_{i}$ occur with equal prob

(detailed balance). If $\mathcal{E}_0 \equiv 0$ no transitions

occur. Quantum field $\Rightarrow$ $|i\rangle \to |f\rangle$ is

possible if $i > f$ and $\mathcal{E}_0 = 0$. Vacuum stimulates

emission.

In dipole approximation $e^{i\vec{k}\cdot\vec{r}} \simeq 1$ and 2.130 $\Rightarrow$

$$\boxed{\hat{\vec{\mathcal{E}}}(\vec{r},t) \approx \hat{\vec{\mathcal{E}}}(t) = i\,\mathcal{E}_0\,\vec{e}\,[\underbrace{\hat{a}e^{-i\omega t}}_{\hat{a}(t)} - \underbrace{\hat{a}^{+}e^{+i\omega t}}_{\hat{a}^{+}(t)}]}$$

where $\mathcal{E}_0 = \sqrt{\dfrac{\hbar\omega}{2\epsilon_0 V}}$ and $\vec{\mathcal{E}_0} \equiv i\,\mathcal{E}_0\,\vec{e}$

This is Heisenberg picture where $\hat{H}^I = \hat{H}^I_H(t)$

Recall (Merzbacher) $\hat{O}_H(t=0) = \hat{O}_S$ for any

Hermitian operator

$$\Rightarrow \boxed{\hat{\vec{\mathcal{E}}} = \vec{\mathcal{E}}_0\,[\hat{a} - \hat{a}^{+}]}$$ In Schrödinger

$$\hat{H}_0 = \underbrace{\hat{H}_{atom}}_{\frac{\hat{p}^2}{2m} - \frac{e^2}{r}} + \underbrace{\hat{H}_{Field}}_{\hbar\omega\,\underbrace{\hat{a}^{+}\hat{a}}_{\text{normal ordered}}} \Rightarrow \text{atom } \underline{\text{uncoupled}} \text{ to field}$$

— no vacuum

$\hat{H} = \hat{H}_0 + \hat{H}_I \leftarrow$ atom-field coupling

$$\hat{H}_I \equiv -\hat{\vec{d}} \cdot \hat{\vec{\varepsilon}}$$

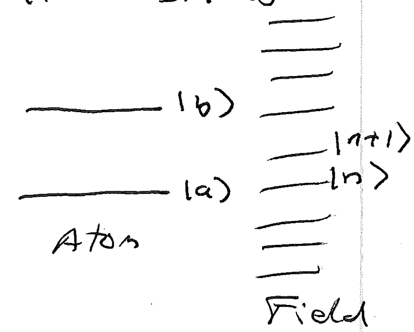
$$= \boxed{-\vec{\varepsilon}_0 \cdot \hat{\vec{d}}\,(\hat{a} - \hat{a}^+)}$$

Hilbert space is tensor product of atom states

$|a\rangle, |b\rangle$ S.T. $\hat{H}_{atom}|a\rangle = E_a|a\rangle$

$\hat{H}_{atom}|b\rangle = E_b|b\rangle$

$\hat{H}_{field}|n\rangle = n\hbar\omega|n\rangle$




with $\hat{H}_I$ atom states now coupled to field states. Thence $|i\rangle = |a\rangle\underset{\text{n photons}}{|n\rangle}$

$\uparrow$ ~~Ground~~ initial state

$\boxed{E_i = E_a + n\hbar\omega}$

$|i\rangle = |a\rangle|n\rangle \rightarrow |f_1\rangle = |b\rangle|n-1\rangle$ emission 1 photon

$\boxed{E_{f_1} = E_b + (n-1)\hbar\omega}$

$|i\rangle = |a\rangle|n\rangle \rightarrow |f_2\rangle = |b\rangle|n+1\rangle$ absorbtion 1 photon

$\boxed{E_{f_2} = E_b + (n+1)\hbar\omega}$

Since 1 photon carries $1\hbar$ angular momentum parity conserved unless. $\hat{\vec{d}}_{if}$

$$H^I_{f_1 i} \equiv \langle f_1 | \hat{H}^I | i \rangle$$

$$= \langle b | \langle n-1 | \left[ -\vec{\varepsilon}_0 \cdot \hat{\vec{d}}\,(\hat{a} - \hat{a}^+) \right] | n \rangle | a \rangle$$

$$= -\vec{\varepsilon}_0 \langle b | \hat{\vec{d}} | a \rangle \langle n-1 | \hat{a} - \hat{a}^+ | n \rangle$$

$$= -\vec{\varepsilon}_0 \cdot \vec{d}_{ba} \left[ \underbrace{\langle n-1 | \hat{a} | n \rangle}_{\sqrt{n}\,\langle n-1 | n-1 \rangle} - \underbrace{\cancel{\langle n-1 | \hat{a}^+ | n \rangle}}_{\sqrt{n+1}\,\langle n-1 | n+1 \rangle} \right]$$

$$\boxed{H^{I}_{f_1 i} = -\vec{\mathcal{E}}_0 \cdot \vec{d}_{ba} \sqrt{n}}$$ absorption

similarly

$$H^{I}_{f_2 i} = \langle f_2 | \hat{H}^{I} | i \rangle$$
$$= \langle b | \langle n+1 | [-\vec{\mathcal{E}}_0 \cdot \hat{\vec{d}} (\hat{a} - \hat{a}^{+})] | n \rangle | a \rangle$$
$$= -\vec{\mathcal{E}}_0 \cdot \langle b | \hat{\vec{d}} | a \rangle \langle n+1 | \hat{a} - \hat{a}^{+} | n \rangle$$
$$= -\vec{\mathcal{E}}_0 \cdot \vec{d}_{bd} [ \sqrt{n} \cancelto{0}{\langle n+1 | n-1 \rangle} - \sqrt{n+1} \langle n+1 | n+1 \rangle ]$$

$$\boxed{H^{I}_{f_2 i} = \sqrt{\underset{\text{stimulated}}{n} + \underset{\text{spontaneous}}{1}} \; \vec{\mathcal{E}}_0 \cdot \vec{d}_{ba}}$$ emission

Loosely the probability of absorbtion and emission are

$$\boxed{\begin{aligned} \tfrac{1}{t} P_{f_1 i} &= \tfrac{1}{t} | H^{I}_{f_1 i} |^2 = \tfrac{1}{t} | \vec{\mathcal{E}} \cdot \vec{d}_{ba} |^2 (n) = W_{f_1 i} \\ \tfrac{1}{t} P_{f_2 i} &= \tfrac{1}{t} | H^{I}_{f_2 i} |^2 = \tfrac{1}{t} | \vec{\mathcal{E}} \cdot \vec{d}_{ba} | (\underset{\text{stimulated}}{n} + \underset{\text{spontaneous}}{1}) = W_{f_2 i} \end{aligned}}$$

absorption <u>rate</u>

emission <u>rate</u>

Hence

Stimulated Emission $\propto n$ ] Reciprocal / Classical
Stimulated Absorption $\propto n$ ] Detailed Balance

spontaneous Emission $\propto 1$ ] occurs when $n = 0$!

Stimulated processes proportional to Intensity.

Spontaneous emission occurs in Vacum.

This is why an excited atom in <u>empty</u> space radiates! Classical Analog: Radiation Reaction Theory.

Quantum: Vacuum Fluctuations "stimulate" Decay.

Einstein first pointed $W_{f_1 i} \neq W_{f_2 i}$ if Planck B.B. is correct. Added spontaneous emission in by hand 1905

$$\frac{W^{\text{emission}}_{f_2 i}}{W^{\text{absorbtion}}_{f_1 i}} = \frac{n+1}{n} \xrightarrow[n\to\infty]{} 1$$

For large $n \to \infty$ we recover classical reciprocity.

Let $\hbar\omega_a = E_a$ and $\hbar\omega_b = E_b$ Atomic $\quad E_n = \hbar\omega_n = n\hbar\omega$

$\hbar\omega = E_0$ field $\Rightarrow E_n^{\pm} \equiv \hbar\omega_n^{\pm} \equiv \hbar(n\pm1)\omega$

with only $\underbrace{|a\rangle, |b\rangle}_{\text{atom}}, \underbrace{|n\rangle}_{\text{field}}$ we have in general

$|b\rangle$

$|a\rangle$

$|n\rangle$

$$|\psi(t)\rangle = |\psi_{atom}\rangle|\psi_{field}\rangle$$

$$= C_i(t)\,|a\rangle|n\rangle\, e^{-i(\omega_a+\omega_n)t}$$

$$+ C_{f_1}(t)\,|b\rangle|n-1\rangle\, e^{-i(\omega_b+\omega_n^+)t} \qquad \underline{\text{ABS}}$$

$$+ C_{f_2}(t)\,|b\rangle|n+1\rangle\, e^{-i(\omega_b+\omega_n^-)t} \qquad \underline{\text{EMIS}}$$

General Solution:

If B.C. $|\psi(0)\rangle = |a\rangle|n\rangle \Rightarrow C_i(0) = 1 \;;\; C_{f_{12}}(0) = 0$

we use Eq. 4.28

$$\boxed{\begin{aligned} C_{f_1}(t) &= -i\frac{1}{\hbar}\int_0^t dt'\, H^I_{f_1 i}\, e^{i\omega_{f_1 i}t'} \\ C_{f_2}(t) &= -i\frac{1}{\hbar}\int_0^t dt'\, H^I_{f_2 i}\, e^{i\omega_{f_2 i}t'} \end{aligned}}$$

$$\omega_{f_1 i} = \omega_{f_1 i} = (\omega_b + \omega_n^-) - (\omega_a+\omega_n) = \omega_{ba} + \omega = \Omega^+_{ba}$$

$$\omega_{f_2 i} = (\omega_3 + \omega_n^+) - (\omega_b + \omega_n) = \omega_{ba} - \omega = \Omega^-_{ba}$$

$$C^{(1)}_{fi} \equiv C^{(1)}_{f_1 i} + C^{(1)}_{f_2 i}$$

$$= -i\gamma\,(\vec{d}_{ba}\cdot\vec{\mathcal{E}}_0)\left\{\int_0^t dt'\left[-\sqrt{n}\,e^{i\Omega^+_{ba}t'} + \underset{\substack{\uparrow\\ \text{new}}}{\sqrt{n+1}}\;e^{i\Omega^-_{ba}t'}\right]\right\}$$

Typo in book

$$= \cancel{i}\gamma\,(\vec{d}_{ba}\cdot\vec{\mathcal{E}}_0)\left\{\sqrt{n+1}\left(\frac{e^{i\Omega^+_{ba}t}-1}{i\Omega^+_{ba}}\right) - \sqrt{n}\left(\frac{e^{i\Omega^-_{ba}t}-1}{\cancel{i}\Omega^-_{ba}}\right)\right.$$

$$= \boxed{\gamma\,(\vec{d}_{ba}\cdot\vec{\mathcal{E}}_0)\left[\underbrace{\sqrt{n+1}\left(\frac{e^{i\Omega^+t}-1}{\Omega^+}\right)}_{\text{Emission}} - \underbrace{\sqrt{n}\left(\frac{e^{i\Omega^-t}-1}{\Omega^-}\right)}_{\text{absorption}}\right]}$$

If $|b\rangle > |a\rangle$ $\quad \frac{\overline{\quad}\,|b\rangle}{\overline{\quad}\,|a\rangle}$ then $\omega_{ba} > 0 \Rightarrow \Omega^+$ is counterrotating and can be dropped. $\Omega^- \to \Delta$

If $|b\rangle < |a\rangle$ $\quad \frac{\overline{\quad}\,|a\rangle}{\overline{\quad}\,|b\rangle}$ then $\omega_{ba} < 0 \Rightarrow \Omega^-$ is counter rotating and dropped with $\Omega^+ \to -|\omega_{ba}| + \omega = -\Delta$

$b>a$ $\quad W_{fi} = \frac{1}{t}P_{fi} = \frac{1}{t}|C_{fi}|^2 = 4\gamma\,|\vec{d}_{ba}\cdot\vec{\mathcal{E}}_0|^2(n)\,\sin^2(\Omega t/2)/\Omega^2 t$

$$= \boxed{\gamma^2\,|\vec{d}_{ba}\cdot(2\sqrt{n}\,\vec{\mathcal{E}}_0)|^2\,\frac{\sin^2(\Delta t/2)}{\Delta^2 t}}$$ which agrees

$|b\rangle$, $|a\rangle$, ABS

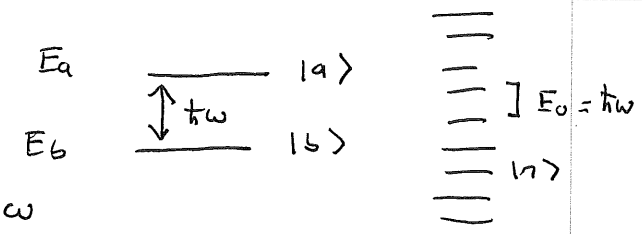
with semiclassical if Eq 4.34 $(2\underset{\substack{\uparrow\\ \text{amplitude}}}{\sqrt{n}}\,\vec{\mathcal{E}}_0)_Q \equiv (\vec{\mathcal{E}}_0)_c$

$b<a$ $$\boxed{W_{fi} = \gamma^2\,|\vec{d}_{ba}\cdot(2\sqrt{n+1}\,\vec{\mathcal{E}}_0)|^2\,\frac{\sin^2(\Delta t/2)}{\Delta^2 t}}$$

New feature emission rate is non zero even if $n=0$

$n \quad 1 \quad |a\rangle$, $|b\rangle$ $\quad$ Both stimulated $n>0$ and spontaneous $n=0$

Suppose a collection of two level atoms are resonant $\Delta = 0$ with $\omega_{field} = \omega_{ab} = \omega$




Let $N_a \equiv$ # atoms in $|a\rangle$ (population)

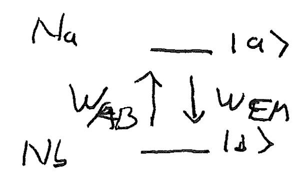
$N_b \equiv$ # atoms in $|b\rangle$ "

Let $W_{em}^{EM}$ = emission rate $= P_{em}/t \quad \propto (n+1)$

$W_{abs}^{AB}$ = absorption rate $= P_{abs}/t \quad \propto (n)$

The <u>rate</u> <u>equations</u> are $N_a = N_a(t)$ ; $N_b = N_b(t)$

$$\boxed{\begin{aligned} \frac{dN_a}{dt} &= -W_{EM} N_a + N_b W_{ABS} \quad \text{(II)} \\ \frac{dN_b}{dt} &= -W_{AB} N_b + W_{EM} N_a \quad \text{(I)} \end{aligned}}$$




Note $\frac{W_{EM}}{W_{AB}} = \frac{n+1}{n}$ so there is an <u>aysemmty</u>.

Einstein now assumes <u>thermal</u> equilibrium

$\Rightarrow \quad \dot{N}_a = \dot{N}_b \equiv 0.$

$\Rightarrow \quad N_a W^{EM} = N_b W^{ABS} \quad \Rightarrow \quad \boxed{\frac{N_b}{N_a} = \frac{W^{EM}}{W^{ABS}} = \frac{\overset{\text{S.E.}}{\downarrow}}{}\!\!\frac{n+1}{n}}$

But from thermo $\frac{N_b}{N_a} = e^{(E_a - E_b)/kT} \equiv e^{\hbar\omega/kT}$ ||

solve $\frac{n+1}{n} = e^{\hbar\omega/kT} \Rightarrow \boxed{n = \bar{n} = \frac{1}{e^{\hbar\omega/kT} - \underset{\uparrow \text{S.E.}}{1}}}$ Planck!

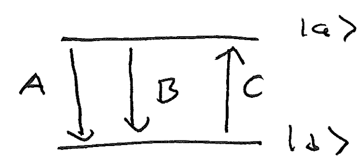
Hence spontaneous emission / vacuum / rad. react is required to get Planck!

for $n >> 1$ $\quad \frac{n+1}{n} \approx 1 \Rightarrow$ Boltzman.

Einstein did not have quantum optics

Einstein used spectral energy $U(\omega) \propto I \propto n$

Introduced $A, B, C$




and explicitly states $A$ is spontaneous rate independent of $U(\omega)$. $B$ is stimulated emission, $C$ absorption

$$\dot{N}_a = -[\overset{SE}{A} + \overset{StE}{BU}]N_a + \overset{SA}{CU}N_b$$

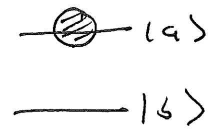
$$\dot{N}_b = -\underset{SA}{CUN_a} + [A + BU]N_a$$

again $\dot{N}_a = \dot{N}_b = 0$

$$\Rightarrow \quad U(\omega) = \frac{A}{Ce^x - B} \qquad x = \hbar\omega/kT$$

To recover Planck $\quad C = B$ and $\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}$

Laser Rates




Let us assume $N_a^{(0)} = N \;\; ; \;\; N_b^{(0)} = 0$

called <u>inversion</u>. All atoms up!

$$\dot{N}_a = -W_E N_a + N_b W_A \quad ; \quad \dot{N}_b = -W_A N_b + W_E N_a$$

$$\Rightarrow \quad \ddot{N}_a = -W_E \dot{N}_a + [-W_A N_b + W_E N_a]$$

For short times $\left\{ \begin{matrix} N_b \cong 0 \\ N_a = N \end{matrix} \right\}$ since no population in $|b\rangle$

$$\Rightarrow \quad \ddot{N}_a \approx -W_E(\dot{N}_a \mp N_a)$$

No role in dynamics

$$\Rightarrow \quad \ddot{N}_a + W_E \dot{N}_a \cancel{- W_E N_a} \cong 0$$

Also

$$\Rightarrow \quad \dot{N}_a = -W_E N_a$$

$$\Rightarrow \quad \boxed{N_a(t) = e^{-g(n+1)t} N_a(0)}$$

$g = 4\gamma^2 |\vec{d}_{ba} \cdot \vec{\mathcal{E}}_0|^2 =$ <u>gain</u> laser gain

Atoms decay from $|a\rangle \to |b\rangle$ at a rate proportional to $(n+1)$.

The more photons in the field the more quickly atoms decay $\Rightarrow$ photons accumulate Exp. <u>Fast</u>. Light Amp. Stim. Emiss.