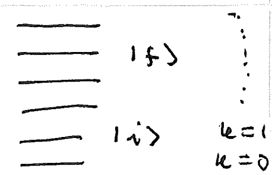
Classical Field

$\vec{k}\cdot\vec{r} \ll 1$ Dipole

$\vec{E} = \vec{E}_0 \cos\omega t$



Assume at $t=0$ atom

has eigenstates $\hat{H}_0 |k\rangle = E_k |k\rangle$

$t \geq 0$ $\mathcal{E}$ field is turned on <u>Gen sol TDSE</u>

$$|t\rangle \equiv |\Psi(t)\rangle = \sum_k C_k(t)\, e^{-\frac{i}{\hbar}E_k t} |k\rangle$$

$$1 = \langle \Psi(0)|\Psi(0)\rangle = \langle \Psi(t)|\Psi(t)\rangle = 1 = \sum_k |C_k(t)|^2$$

$|C_k(t)|^2$ is prob atom is in state $|k\rangle$ at time t.

$$i\hbar |\dot{t}\rangle = \hat{H}|t\rangle \qquad \text{T.D.S.E.}$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \sum_k C_k(t)\, e^{-\frac{i}{\hbar}E_k t} |k\rangle = \left(\hat{H}_0 + \hat{H}_I\right) \sum_k C_k(t)\, e^{-\frac{i}{\hbar}E_k t} |k\rangle$$

$$\Rightarrow i\hbar \sum_k \left\{\left[\dot{C}_k - \frac{i}{\hbar}E_k C_k\right] e^{-\frac{i}{\hbar}E_k t} |k\rangle\right. = \sum_k C_k\left(E_k + \hat{H}_I\right) e^{-\frac{i}{\hbar}E_k t} |k\rangle$$

$$\langle \ell | e^{+\frac{i}{\hbar}E_\ell t} \rightarrow \quad \Rightarrow i\hbar \dot{C}_\ell + \cancel{E_\ell C_\ell} = \cancel{C_\ell E_\ell} + \sum_k C_k \langle \ell | \hat{H}_I | k\rangle e^{i\omega_{\ell k} t}$$

$$\Rightarrow \boxed{\dot{C}_\ell = -\frac{i}{\hbar} \sum_k C_k \langle \ell | \hat{H}_I | k\rangle e^{i\omega_{\ell k} t}} \qquad \omega_{\ell k} = \omega_\ell - \omega_k$$

Eq. 4.20

This gives us a first-order differential equation or, an entire <u>infinite</u> set of coupled differential equations. For realistic problem is to decide which subset of the states $|k\rangle$ are important and truncate the sum at some large $K = k$ and then solve this numerically.

To get an analytic expression we use time-dependent perturbation theory (TDPT). Typically we assume at $t=0$ that all the $C_\ell(t) \equiv 0$ except one $C_{initial}(0)$. That is $\exists\ \ell = i$ s.t. $C_\ell(t=0) = C_i$ and $C_{\ell \neq i}(0) = 0$.

As time evolves the prob. to find the electron in some other state $C_{final}(t)$ is nonzero

$P_{if}(t) \equiv |C_f(t)|^2$ where the relevant matrix element for the transition is

$$\langle f | \hat{H}_I | i \rangle = \langle f | (-\hat{\vec{d}} \cdot \vec{E}(t)) | i \rangle = \hat{H}^I_{fi}$$

If we assume that $|H^I_{fi}|^2 \ll E_{fi} \equiv \hbar\omega_f - \hbar\omega_i$ that is the coupling of the atom to field is weak we can use TDPT. This is in Merzbacher.

$$0 \leq \lambda \leq 1$$

$$\Rightarrow \quad C_\ell(t) = \sum_m \lambda^m C_\ell^{(m)}(t)$$

where this is a power series in λ with coefficients $C_\ell^{(m)}(t)$.

with $C_k(t) = \sum_n \lambda^n C_k^{(n)}(t)$ we can sub in 4.20 but also take

$$\langle \ell | \hat{H} | k \rangle = H_{\ell k} \rightarrow \lambda H_{\ell k}$$

$$\frac{\partial}{\partial t}\left[\sum_m \lambda^m C_\ell^{(m)}\right] = -\frac{i}{\hbar}\sum_{kn}\lambda^n C_k^{(n)}\lambda\langle \ell|\hat{H}|k\rangle e^{i\omega_{\ell k}t}$$

$$\Rightarrow \sum_m \lambda^m \dot{C}_\ell^m = -\frac{i}{\hbar}\sum_{kn}\lambda^{n+1}C_k^n H_{\ell k} e^{i\omega_{\ell k}t} = -\frac{i}{\hbar}\sum_{km}\lambda^{m+1}C_k^m H_{\ell k} e^{i\omega_{\ell k}t}$$

we may equate powers of λ in $\sum_m$

$\lambda^0 \Rightarrow$ $\boxed{\dot{C}_\ell^0 = 0}$ The λ^{m+1} implies no zeroth term

$\lambda^1 \Rightarrow$ $\dot{C}_\ell^1 = -\frac{i}{\hbar}\sum_k C_k^0 H_{\ell k} e^{i\omega_{\ell k}t}$ $\qquad C^{(m)} \equiv C^m$

$\lambda^2 \Rightarrow$ $\dot{C}_\ell^2 = -\frac{i}{\hbar}\sum_k C_k^1 H_{\ell k} e^{i\omega_{\ell k}t}$ $\qquad H^I \equiv H$

$\vdots \qquad \vdots$

$\lambda^n \Rightarrow$ $\dot{C}_\ell^n = -\frac{i}{\hbar}\sum_k C_k^{n-1} H_{\ell k} e^{i\omega_{\ell k}t}$

So the idea is to solve the series by plugging in $C_\ell^{(n)}$ to get $C_\ell^{(n+1)}$. If we keep all ∞ therms in series $\sum_{m=0}^{\infty}\lambda^n C_\ell^{(m)} = C_\ell$ the solution is exact. Typically we truncate at $m = 2$ or 3.

Taking λ^1 we get to integrate and

$$C_\ell^1(t) = -\frac{i}{\hbar}\sum_k \int_0^t dt' \, H(t')_{\ell k} e^{i\omega_{\ell k}t}$$

~~where $\lambda^0 \Rightarrow$ WLOG $C_k^{(0)}(t) = 1 \ \forall k$~~ (OK)

Then if we take $|k\rangle \to |i\rangle = |\text{initial}\rangle$; $|\ell\rangle \to |f\rangle = |\text{final}\rangle$

if H^I_{fi} is small then at $t=0$

$C_i(0) = 1$ and $C_{f\neq i}(0) = 0$ for $t > 0$

$$\boxed{|C_i(t)| \simeq 1 \qquad |C_{f\neq i}(t)| \lll 1}$$

Under this two level approximation we assume $\omega \sim \omega_{fi}$

and only terms $|k\rangle = |i\rangle$ and $|k\rangle = |f\rangle$ contribute

$$\boxed{\dot{c}_f^{(1)}(t) = -iy \int_0^t dt'\, H^{\pm}_{fi}(t')\, e^{i\omega_{fi} t}}$$

First order Eq.

Insert into λ^2 to get

$$\boxed{\dot{c}_f^{(2)}(t) = (-iy)^2 \int_0^t \int_0^{t'} dt'dt'' \sum_l H_{fl}(t')\, H_{li}(t'')\, e^{i\omega_{fl} t'} e^{i\omega_{li} t''}}$$

Recall that $\langle f | \hat{d} | i \rangle = e \int dr^3\, \Psi_f^* \vec{r}\, \Psi_i = 0$

if $f = i$ (see Merzbacher) that is $\hat{d}$ connects

states of opposite parity $\Rightarrow$ $H^{\pm}_{ii} = 0$ so that

first order correction is

$$\boxed{c_f^{(1)}(t) = -iy \int_0^t dt'\, H^{\pm}_{fi}(t')\, e^{i\omega_{fi} t'}}$$

Let $H^I = -\hat{\vec{d}} \cdot \vec{E}_0 \cos\omega t$ the driving field at ω.

We can integrate letting (semiclassical) $\langle f | \hat{\vec{d}} | i \rangle = \vec{d}_{fi}$

$$c_f^{(1)}(t) = +iy\left(\vec{E}_0 \cdot \vec{d}_{fi}\right) \int_0^t \frac{e^{i\overbrace{(\omega+\omega_{fi})}^{\Omega_+} t'} + e^{-i\overbrace{(\omega-\omega_{fi})}^{\Delta} t'}}{2}$$

$$= \boxed{\frac{iy}{2}\left(\vec{E}_0 \cdot \vec{d}_{fi}\right) \left\{ \cancel{\frac{e^{i\Omega_+ t} - 1}{(i\Omega_+)}} + \frac{e^{-i\Omega_- t} - 1}{(-i\Omega_-)} \right\}}$$

RWA

If $\omega_{fi} > 0$ and $\Delta \equiv \omega - \omega_{fi} \approx 0$ the Ω_- term is in resonance and the Ω_+ is out of resonance so we drop it. Δ corotates with atom frequency so Rotating Wave Approx. (RWA).

Hence $C^{(1)}_{fi}(t) \cong -\overbrace{\frac{q}{2}(\vec{E}_0\cdot\vec{d}_{fi})}^{-U^0_{fi}} \frac{e^{-i\Delta t}-1}{\Omega\,-}$

$$= -\frac{q}{2}(\vec{E}_0\cdot\vec{d}_{fi})\, e^{-i\Delta t/2}\, 2i\left[\frac{e^{-i\Delta t/2}-e^{i\Delta t/2}}{2i\Delta}\right]$$

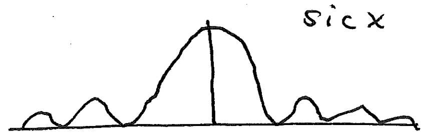
$$= \frac{q\, i}{\cancel{2}}\, U^0_{fi}\, e^{-i\Delta t/2}\, \frac{\sin(\Delta t/2)}{\Delta}$$

$$\Rightarrow \quad P^{(1)}_{fi}(t) \equiv |C_{fi}(t)|^2 = \boxed{q^2\, |\vec{E}_0\cdot\vec{d}_{fi}|^2\, \frac{\sin^2(\Delta t/2)\, t^2}{(\Delta t/2)^2\, 4}}$$

To first order in λ' this is prob electron is in state $|f\rangle$ at time $t>0$.

Let $\text{sinc}(x) \equiv \frac{\sin x}{x}$

sicx



x

Note $\text{sinc}(0) = 1$ is maximum at $x=0$ L'Hospital $\frac{\sin x}{x} \to \frac{\cos x}{1} \to 1$ at $x \to 0$

$$P^{(1)}_{fi}(t) = \boxed{\frac{1}{4} q^2 U^2_{fi}\, \text{sinc}^2(\Delta t/2)\, t^2}$$

$$P^{(1)}_{fi}(0) = 0$$

Note: $\Delta \to 0 \Rightarrow$ $\boxed{P^{(1)}_{fi}(t)\Big|^{\Delta=0}_{max} = \frac{q^2}{4} U^2_{fi}\, t^2}$ Resonance $\Delta=0$

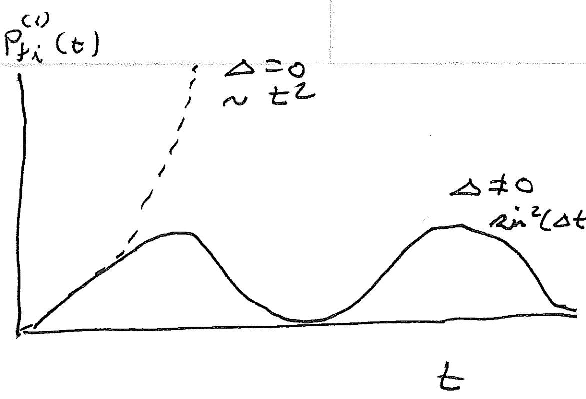
Now for $\Delta \neq 0$ $P^{(1)}_{fi}(t) \propto \frac{\sin^2(\Delta t/2)}{\Delta^2}$

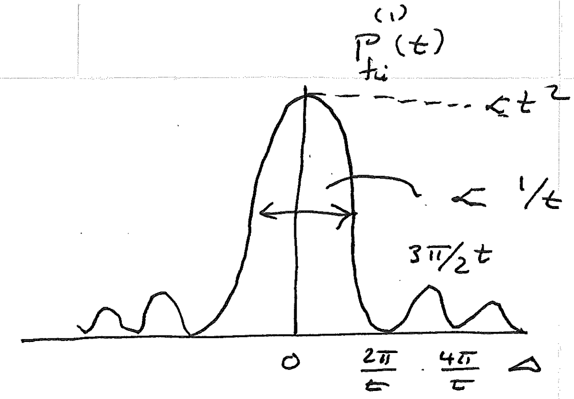
which is periodic with maxima when $\Delta t/2 = \pi/2$

or $\boxed{t = \pi/\Delta}$ atomic resonance beats on field ω

with beat period $T = 2\pi/\Delta$ maximum are

$$\boxed{P^{(1)}_{fi}(t)\Big|^{\Delta\neq 0}_{MAX} = \frac{1}{4\hbar^2} U^2_{fi} \frac{1}{\Delta^2} \Big|_{t=2\pi m/\Delta 2}}$$





From this we can see maximum prob at $\Delta = 0$

For $\Delta \neq 0$ probability oscillates periodically from $|i\rangle \to |f\rangle \to |i\rangle$. These are Rabi oscillations

There are oscillations when $\Delta = 0$ but this theory can't handle it. We can see that as a fcn of Δ $P(t)$ is mostly peaked at $\Delta = 0$ with height t^2 and width $1/t$ so area $\approx (t^2)(1/t) \approx t$.

More Precisely

$$\int_{-\infty}^{\infty} d\Delta \; P(\Delta, t) = \int \frac{1}{4\hbar^2} U^2 \frac{\sin^2(\Delta t/2)}{(\Delta t/2)^2} t^2 \left(\frac{d\Delta\, t}{2}\right) \frac{2}{t}$$

$$= \int_{-\infty}^{\infty} \frac{2t}{4\hbar^2} U^2 \frac{\sin^2 x}{x^2} = \frac{t U^2}{2\hbar^2} \pi = \boxed{\frac{\pi U^2}{2\hbar^2} t}$$

The area of the peak plus wiggles.

Using the sinc representation of Dirac δ

$$\boxed{\lim_{t \to \infty} \frac{\sin^2(\Delta t/2)}{\Delta^2} \approx \frac{\pi}{2} t \, \delta(\Delta)}$$

Hence for $t >> 0$

$$\boxed{P^{(1)}_{fi}(t) \approx \frac{\pi}{2\hbar^2} U_{fi}^2 \, t \, \delta(\Delta_{fi})}$$

$\Delta_{fi} = \omega - \omega_{fi}$

The rate is defined $\boxed{W_{fi} \equiv P_{fi}/t}$

$$\boxed{W^{(1)}_{fi} = \frac{\pi}{2\hbar^2} U^2_{fi} \, \delta(\Delta_{fi})}$$

which is indepent of t

We did all this assuming only one transition $|i\rangle \to |f\rangle$ participates. If a collection of final states $\{|f\}\rangle$ participates then

$$\boxed{W_{\{f\}i} = \frac{\pi}{2\hbar^2} \sum_{\{f\}} U^2_{fi} \, \delta(\Delta_{fi})}$$

This is called Fermi's Golden Rule! Remember we assumed only a single mode $\vec{E}$ for free space $\delta(\Delta_{fi})$ becomes $\rho(\omega_{fi})$ the density of modes.

For thermal light, single mode, then

$$U^2_{fi} \to F(\omega) \equiv |\langle f| \vec{d} \cdot \vec{E}_0(\omega) |i\rangle|^2$$

since thermal light has random phase we can treat it as an independent set of <u>drivers</u>

$$\boxed{W_{fi} \to \frac{P^{(1)}_{fi}(t)}{t} = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} d\omega \, \frac{\sin^2[\Delta_{fi}(\omega) t/2]}{\Delta^2_{fi}(\omega)} F(\omega)}$$

since $\sin \sim \delta(\Delta)$ we can pull $F(\omega) \to F(\omega_{fi})$ out of integral provided $E(\omega)$ is broad band.

$$\Rightarrow W_{fi} = \frac{\pi}{2\hbar^2} F(\omega_{fi}) = \boxed{\frac{\pi}{2\hbar^2} |\vec{E}_0(\omega_{fi}) \cdot \vec{d}_{fi}|^2}$$

Thermal Light