Coherent states are normal $\langle\alpha|\alpha\rangle = 1$ but <u>not</u> orthogonal! $\langle\alpha|\beta\rangle \neq 0$ if $|\alpha\rangle \neq |\beta\rangle$

$$|\alpha\rangle = \sum_n e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\beta\rangle = \sum_m e^{-\frac{|\beta|^2}{2}} \frac{\beta^m}{\sqrt{m!}} |m\rangle$$

$$\langle\beta| = \{|\beta\rangle\}^\dagger = \sum_m e^{-\frac{|\beta|^2}{2}} \frac{\beta^{*m}}{\sqrt{m!}} \langle m|$$

$$\langle\beta|\alpha\rangle = e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} \sum_{mn} \frac{\beta^{*m}\alpha^n}{\sqrt{m!\,n!}} \overbrace{\langle m|n\rangle}^{\delta_m^n}$$

$$= e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} \sum_n \frac{(\beta^*\alpha)^n}{n!} \longleftarrow \underline{\exp}$$

$$= e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} e^{\beta^*\alpha}$$

$$= e^{-\frac{1}{2}\left[|\alpha|^2 - 2\beta^*\alpha + |\beta|^2\right]}$$

Noting: $|\beta-\alpha|^2 \equiv (\beta-\alpha)^\dagger(\beta-\alpha) = (\beta^*-\alpha^*)(\beta-\alpha)$

$$= |\beta|^2 - \alpha^*\beta - \beta^*\alpha + |\alpha|^2$$

$$= |\alpha|^2 + |\beta|^2 + \underbrace{2\beta^*\alpha - 2\beta^*\alpha}_{=0} - \alpha^*\beta - \beta^*\alpha$$

complete the square

$$= |\alpha|^2 - 2\beta^*\alpha + |\beta|^2 + \beta^*\alpha - \alpha^*\beta$$

$$\Rightarrow |\alpha|^2 - 2\beta^*\alpha + |\beta|^2 = |\beta-\alpha|^2 + \alpha^*\beta - \beta^*\alpha$$

$$\Rightarrow \boxed{\langle\beta|\alpha\rangle = e^{\frac{1}{2}[\beta^*\alpha - \alpha^*\beta]}\, e^{-\frac{1}{2}|\beta-\alpha|^2}}$$

$\alpha = \alpha' + i\alpha''$ $\quad \beta = \beta' + i\beta''$ $\quad z = \beta^*\alpha = x + iy$

$\alpha^* = \alpha' - i\alpha''$ $\quad \beta^* = \beta' - i\beta''$ $\quad \Rightarrow z^* = \alpha^*\beta = x - iy$

$\Rightarrow z - z^* = 2iy$

$\Rightarrow \quad \beta^*\alpha - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha^*\beta)$

$\Rightarrow$

$$\langle\beta|\alpha\rangle = e^{i\,\mathrm{Im}(\alpha^*\beta)}\; e^{-\frac{1}{2}|\beta-\alpha|^2}$$

$$\langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle = e^{-i\,\mathrm{Im}(\alpha^*\beta)}\; e^{-\frac{1}{2}|\beta-\alpha|^2}$$

$$\Rightarrow \boxed{|\langle\beta|\alpha\rangle|^2 = \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle = e^{-|\beta-\alpha|^2} = e^{-|\alpha-\beta|^2}}$$

This is never zero! let $\alpha = |\alpha|e^{i\theta} = \sqrt{\bar{n}}\, e^{i\theta}$

$\beta = |\beta|e^{i\varphi} = \sqrt{\bar{m}}\, e^{i\varphi}$

$$|\alpha-\beta|^2 = (\alpha-\beta)^*(\alpha-\beta)$$

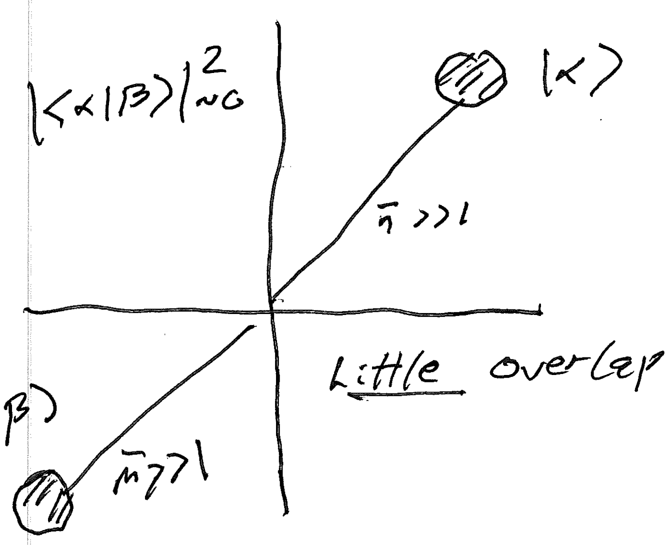
$$= |\alpha|^2 + |\beta|^2 - |\alpha||\beta|e^{i(\theta-\varphi)} - |\alpha||\beta|e^{-i(\theta-\varphi)}$$

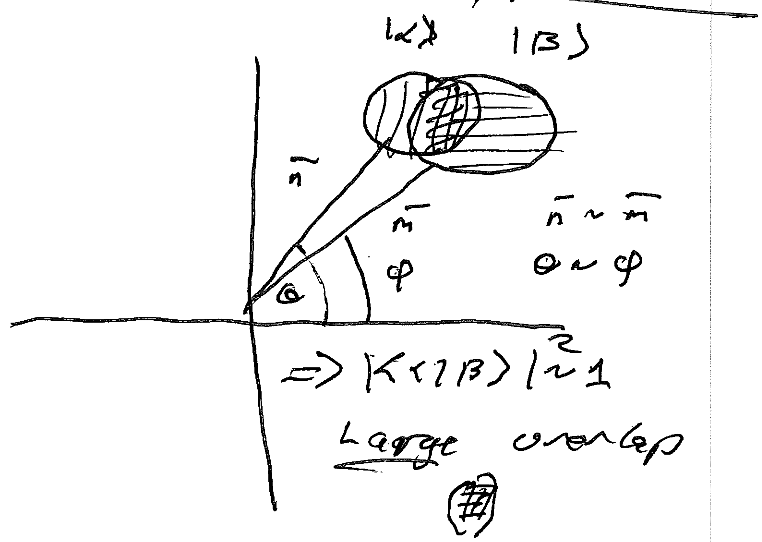
$$= \sqrt{\bar{n}}^{\,2} + \sqrt{\bar{m}}^{\,2} - 2\sqrt{\bar{n}\bar{m}}\cos(\theta-\varphi)$$

$$\Rightarrow |\langle\beta|\alpha\rangle|^2 = e^{-\bar{n}}\, e^{-\bar{m}}\, e^{-2\sqrt{\bar{n}\bar{m}}\cos(\theta-\varphi)}$$

This $\rightarrow 0$ if $\bar{n}\rightarrow\infty$; $\bar{m}\rightarrow\infty$

and $\theta \neq \varphi$ so classical and <u>well</u> <u>separated</u>





$\boxed{\Theta}$ coherent states are over complete

$$\boxed{\frac{1}{\pi}\int d\alpha^2 \; |\alpha\rangle\langle\alpha| = \hat{1}}$$

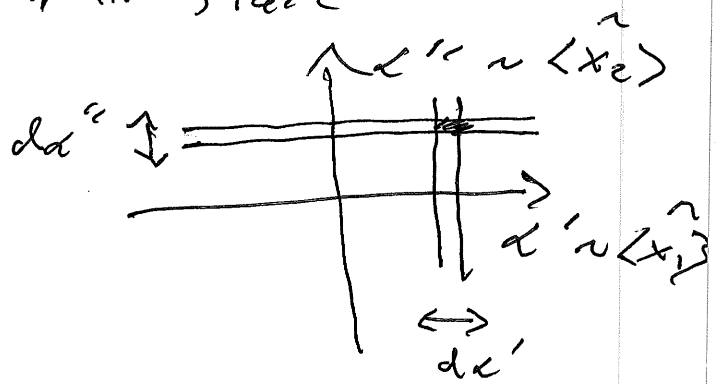
where $d\alpha^* \equiv d\alpha' d\alpha'' = d\,\mathrm{Real}(\alpha)\, d\,\mathrm{Im}(\alpha)$

if $\alpha \in$ Complex i.e. $\alpha = \alpha' + i\alpha''$

with $\alpha', \alpha'' \in$ Real

Double Integral over Complex Space

Proof $\left(\bar{n} \equiv |\alpha|^2\right)$



$$|\alpha\rangle = e^{-\frac{\bar{n}}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle\alpha| = e^{-\frac{\bar{n}}{2}} \sum_m \frac{\alpha^{*m}}{\sqrt{m!}} \langle m|$$

Ket - Bra op : $|\alpha\rangle\langle\alpha| = e^{-\bar{n}} \sum_{nm} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!\,n!}} |n\rangle\langle m|$

$$\int d\alpha^2 \, |\alpha\rangle\langle\alpha| = \int d\alpha^2 \sum_{nm} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!\,m!}} |n\rangle\langle m| \, e^{-\bar{n}}$$

Let $\alpha = \alpha' + i\alpha'' = |\alpha| e^{i\theta} = \sqrt{\bar{n}}\, e^{i\theta} \equiv r e^{i\theta}$

$$\int d\alpha^2 \equiv \int d\alpha' \int d\alpha'' = \iint d\alpha' d\alpha'' \to \int_0^\infty r\,dr \int_0^{2\pi} d\theta$$

switch to polar $r^2 \equiv \bar{n}$

$$\Rightarrow \int d\alpha^2 \, |\alpha\rangle\langle\alpha| = \sum_{nm} \int_0^\infty r\,dr \int_0^{2\pi} d\theta \; r^{n+m} e^{-r^2} e^{i(n-m)\theta} \frac{|n\rangle}{\sqrt{n!}} \frac{\langle m|}{\sqrt{m!}}$$

But $\int_0^{2\pi} d\theta \, \exp[i(n-m)\theta] \equiv 2\pi\, \delta_n^m$

$$\Rightarrow \int d\alpha^2 \, |\alpha\rangle\langle\alpha| = 2\pi \sum_n \left(\int_0^\infty dr \; r^{2n+1} e^{-r^2} \right) \frac{|n\rangle\langle n|}{n!}$$

Let $u \equiv r^2$ $\quad du = 2r\,dr \quad r: 0 \to \infty$

$u: 0 \to \infty$

$$\Rightarrow \quad \int d\alpha^2 |\alpha\rangle\langle\alpha| = \pi \sum_n \left(\int_0^\infty du \; u^n e^{-u} \right) \frac{|n\rangle\langle n|}{n!}$$

D.I.P.-S.H.I.T.

D		I
u^n		e^{-u}
$n u^{n-1}$	$(+)$	
$n(n-1) u^{n-2}$	$(-)$	
$\vdots$		
	$(-)^{n+1}$	
$n!\, u^0$	$(-)^n$	e^{-u}

$$\int_0^\infty u^n e^{-u} = -n!\, e^{-u} \Big|_0^\infty$$

$$= -n! \left[0 - 1 \right] = n!$$

$$\Rightarrow \int d\alpha^2 |\alpha\rangle\langle\alpha| = \pi \sum_n \frac{n!}{n!} |n\rangle\langle n| = \pi \sum_n |n\rangle\langle n| = \mathbb{1} \cdot \pi$$

$$\Rightarrow \boxed{\frac{1}{\pi} \int d\alpha^2 |\alpha\rangle\langle\alpha| \equiv \hat{\mathbb{1}}}$$

$$\frac{1}{\pi}\int d\alpha^2\, |\alpha\rangle\langle\alpha| = \hat{\mathbb{1}} \qquad \text{completeness} \Rightarrow \text{any } |\psi\rangle \in \mathcal{H}_{SHO}$$

can be expanded in a "Fourier" series:

$$|\psi\rangle = \hat{\mathbb{1}}\,|\psi\rangle = \frac{1}{\pi}\int d\alpha^2\, |\alpha\rangle \underbrace{\langle\alpha|\psi\rangle} = \frac{1}{\pi}\int d\alpha^2\, \underbrace{C_\alpha(\psi)}\, |\alpha\rangle$$

where $C_\alpha(\psi) = \langle\alpha|\psi\rangle$ are "Fourier" coefficients.

where $C_\alpha(\psi) = \sum_n e^{-\frac{\bar{n}}{2}} \frac{\alpha^{*n}}{\sqrt{n!}} \langle n|\psi\rangle$

and $P_\alpha(\psi) = |C_\alpha(\psi)|^2 = \sum_{nm} e^{-\bar{n}} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!\,n!}} \langle n|\psi\rangle\langle\psi|m\rangle$

Let $|\psi\rangle = |\beta\rangle$ another coherent state

$$C_\alpha(|\beta\rangle) \equiv \langle\alpha|\beta\rangle = \sum_{nm} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \frac{\alpha^{*m}\beta^n}{\sqrt{m!\,n!}}\langle m|n\rangle$$

$$= e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}} \sum_n \frac{(\alpha^*\beta)^n}{n!} = e^{-\frac{(|\alpha|^2+|\beta|^2)}{2}}\, e^{\alpha^*\beta}$$

$$= \exp\left\{-\tfrac{1}{2}\left[|\alpha|^2 + 2\alpha^*\beta + |\beta|^2\right]\right\}$$

$$= \exp\left\{-\tfrac{1}{2}\left[\alpha^*\alpha + 2\alpha^*\beta + \beta^*\beta\right]\right\}$$

Hence: $|\beta\rangle = \frac{1}{\pi}\int d\alpha^2\, C_\alpha(\beta)\,|\alpha\rangle$

$$= \frac{1}{\pi}\int d\alpha^2 \exp\left[-\tfrac{1}{2}\left[|\alpha|^2 + 2\alpha^*\beta + |\beta|^2\right]\right\}|\alpha\rangle \overset{?}{\neq} |\beta\rangle$$

Compare to: $|m\rangle = \hat{\mathbb{1}}\,|m\rangle = \sum_n |n\rangle\langle n|m\rangle = |m\rangle$

$\{|n\rangle\}$ are independent vectors that span $\mathcal{H}_{SHO} \Rightarrow$ complete

$\{|\alpha\rangle\}$ are <u>dependent</u> vectors that span $\mathcal{H}_{SHO} \Rightarrow$ <u>over</u> complete!

We recover independence only in limit that

$$G \equiv \frac{1}{\pi}\exp\left[-\tfrac{1}{2}\left[|\alpha|^2 + 2\alpha^*\beta + |\beta|^2\right]\right] \longrightarrow \delta(\alpha-\beta)$$

$$\Rightarrow |\beta\rangle = \int d\alpha^2\, \delta(\alpha-\beta)\,|\alpha\rangle = |\beta\rangle$$

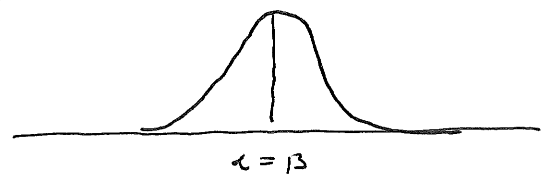
When does this hold?

Note $\quad |\alpha|^2 + \alpha^*\beta + |\beta|^2 = (\alpha^*\alpha + \alpha^*\beta + \underbrace{\beta^*\alpha + \beta^*\beta) - \beta^*\alpha}_{\text{complete } \square \text{ sq}}$

$= |\alpha - \beta|^2 - \beta^*\alpha$

$G = \frac{1}{\pi} \exp\left[-\frac{1}{2}|\alpha-\beta|^2\right] e^{-\beta^*\alpha}$ which is Gaussian

peaked at $\alpha = \beta \quad \sim \quad \delta(\alpha - \beta)$



Let $\quad \alpha \equiv |\alpha| e^{i\theta}$

$\beta \equiv |\beta| e^{i\varphi}$

$$G_{\alpha\beta} = \frac{e^{-\beta^*\alpha}}{\pi} \exp\left[-\frac{1}{2}\left[(|\alpha|e^{-i\theta} - |\beta|e^{-i\varphi})(|\alpha|e^{i\theta} - |\beta|e^{i\varphi})\right]\right]$$

$$= \frac{e^{-\beta^*\alpha}}{\pi} \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2|\alpha||\beta|\cos(\theta - \varphi)\right)\right]$$

Hence $\quad \boxed{G_{\alpha\beta} \sim \delta(\alpha - \beta) \quad \therefore}$

For arbitrary $|\psi\rangle \in \mathcal{H}_{SHO}$ recall

$$C_\alpha(\psi) \equiv \langle\alpha|\psi\rangle = e^{-\frac{\bar{n}}{2}} \sum_n \frac{\alpha^{*n}}{\sqrt{n!}} \underbrace{\langle n|\psi\rangle}_{:= \psi_n}$$

Let $\quad \psi(\alpha^*) \equiv \sum_{n=0}^{\infty} \psi_n \frac{\alpha^{*n}}{\sqrt{n!}}$

or $\quad \forall\, z \in \mathbb{C}$omplex $\quad \boxed{\psi(z) = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{\sqrt{n!}}}$

This is absolutely convergent in Complex Plane

Suppose $F(\hat{a}, \hat{a}^+)$ is any well defined function of $\hat{a}$, $\hat{a}^+$. $\exists$ Taylor's Expansion in 2D s.t.

$$F(\hat{a}, \hat{a}^+) = \sum_{mn} \left[\frac{\partial^m \partial^n}{\partial x^m \partial y^n} F(x,y) \right] \Bigg|_{\substack{x=0 \\ y=0}} \overbrace{\frac{\hat{a}^m}{m!} \frac{\hat{a}^{+n}}{n!}}^{?}$$

Problem: $[\hat{a}, \hat{a}^+] = \hat{\mathbb{1}} \neq 0$ so expansion not well defined for noncommuting ops.

Solution: Let $\hat{F} \equiv F(\hat{a}, \hat{a}^+)$

$$\Rightarrow \hat{F} = \hat{\mathbb{1}} \hat{F} \hat{\mathbb{1}} = \sum_{mn} |m\rangle \underbrace{\langle m | \hat{F} | n \rangle}_{\equiv F_{mn} \text{ matrix elt.}} \langle n| = \sum_{mn} |m\rangle F_{mn} \langle n|$$

$\Rightarrow$ # state basis expansion of $\hat{F}$.

However $\hat{F} = \hat{\mathbb{1}} \hat{F} \hat{\mathbb{1}} = \frac{1}{\pi^2} \iint d\alpha^2 d\beta^2 \, |\beta\rangle \underbrace{\langle \beta | \hat{F} | \alpha \rangle}_{\equiv F_{\beta\alpha}} \langle \alpha|$

$$= \boxed{\frac{1}{\pi^2} \iint d\alpha^2 d\beta^2 \, |\beta\rangle F_{\beta\alpha} \langle\alpha|}$$

The coherent state basis expansion $\neq \hat{F}$!

Given $\{|n\rangle\}$ basis what is xfrm to $\{|\alpha\rangle\}$ basis?

$$F_{\beta\alpha} \equiv \langle \beta | \hat{F} | \alpha \rangle = \langle \beta | \sum_{mn} |m\rangle F_{mn} \langle n | \alpha \rangle$$

$$= \sum_{mn} \langle \beta | m \rangle F_{mn} \langle n | \alpha \rangle$$

$$= \sum_{mn} \left\{ \sum_p e^{-\frac{|\beta|^2}{2}} \frac{\beta^{*p}}{\sqrt{p!}} \langle p | m \rangle \right\} \left\{ \sum_q e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^q}{\sqrt{q!}} \langle n | q \rangle \right\} F_{mn}$$

$$= \sum_{mn} \left\{ e^{-\frac{|\beta|^2}{2}} \frac{\beta^{*m}}{\sqrt{m!}} e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \right\} F_{mn} \qquad \text{Basis Rotation in } \mathcal{H}.$$

$$= e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \left\{ \sum_{mn} \frac{\alpha^n \beta^{*m}}{\sqrt{n! m!}} F_{mn} \right\}$$

$$\equiv e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \left\{ F(\beta^*, \alpha) \right\}$$

If $\hat{F}^{+} = \hat{F}$ hermetian then $F(\beta^*, \alpha)$

is doubly absolutely convergent in $\mathbb{C} \times \mathbb{C}$

Hence <u>now</u> we can expand $F(\beta^*, \alpha)$ instead

of $F(\hat{a}, \hat{a}^{+})$! Let $\langle \beta | \to \langle \alpha |$

Let $\beta^* \to \alpha^*$

$$\Rightarrow \quad \langle \alpha | \hat{F} | \alpha \rangle = e^{-\bar{n}} F(\alpha^*, \alpha) \qquad (\bar{n} \equiv |\alpha|^2)$$

$$\Rightarrow \quad e^{\bar{n}} \underbrace{\langle \alpha | \hat{F} | \alpha \rangle}_{F_{\alpha\alpha}} = F(\alpha^*, \alpha) \equiv \sum_{mn} F_{mn} \frac{\alpha^{*m} \alpha^{n}}{\sqrt{m!\,n!}} \qquad (*)$$

These are diagonal Matrix elts $F_{\alpha\alpha}$ in $|\alpha\rangle$ basis

We <u>now</u> can expand $F(\alpha^*, \alpha)$ instead of $F(\hat{a}, \hat{a}^{+})$

with no problem in Taylor's

$$\boxed{e^{\bar{n}} F(\alpha^*, \alpha) \equiv \sum_{mn} \left[\frac{\partial^m \partial^n F(\alpha^*, \alpha) e^{\bar{n}}}{\partial^m \alpha^* \partial^n \alpha} \right] \Bigg|_{\substack{\alpha^* = 0 \\ \alpha = 0}} \frac{\alpha^{*m}}{m!} \frac{\alpha^{n}}{n!}}$$

compare term by term to $(*)$

$$\Rightarrow \quad e^{\bar{n}} \left[\frac{\partial^m \partial^n}{\partial \alpha^{*m} \partial^n \alpha} F(\alpha^*, \alpha) e^{\bar{n}} \right]_{\substack{\alpha^* = 0 \\ \alpha = 0}} \frac{\alpha^{*m}}{m!} \frac{\alpha^{n}}{n!} \equiv$$

$$\equiv F_{mn} \frac{\alpha^{*m} \alpha^{n}}{\sqrt{m!\; n!}}$$

$$\Rightarrow \quad \boxed{F_{mn} \equiv \frac{1}{\sqrt{m!\,n!}} \left[\frac{\partial^m \partial^n}{\partial^m \alpha^* \partial^n \alpha} F(\alpha^*, \alpha) e^{\alpha^* \alpha} \right]_{\substack{\alpha^* = 0 \\ \alpha = 0}}}$$

XFRM From $|\alpha, \beta\rangle \to |m, n\rangle$ basis