Let $\vec{j}(\vec{r},t)$ be a classical electric current

Let $\hat{\vec{A}}(\vec{r},t) = \vec{e} A_0 [\hat{a} e^{i\varphi} + \hat{a}^+ e^{-i\varphi}]$

where $\hat{\vec{A}}(\vec{r},t)$ is a vector $\hat{op}$ for the E&M Field

and $A_0 \equiv \sqrt{\frac{\hbar}{2\omega\epsilon_0 V}}$ ← volume is the single-photon vector

↑ TYPO

potential amplitude and $\varphi = \varphi(\vec{r},t) \equiv \vec{k}\cdot\vec{r} - \omega t$

From Jackson the classical interaction energy $V(t)$ between

↓ TYPO

current and field is $V(t) = -\int d\tilde{r}^3 \, \vec{j}\cdot\vec{A}$; take $\vec{A} \to \hat{\vec{A}}$

$$\Rightarrow \hat{U}(t) = -\int_V d\vec{r}^3 \, \vec{j}(\vec{r},t)\cdot\hat{\vec{A}}(\vec{r},t)$$

$$= -A_0 \int d\vec{r}^3 \{\hat{a}\,\vec{e}\cdot\vec{j} e^{i\varphi} + \hat{a}^+\vec{e}\cdot\vec{j} e^{-i\varphi}\}$$

$$= -A_0\{\hat{a}\,\vec{e}\cdot[\int d\vec{r}^3 \vec{j} e^{i\vec{k}\cdot\vec{r}}]e^{-i\omega t} + \hat{a}^+\vec{e}\cdot[\int d\vec{r}^3 \vec{j} e^{-i\vec{k}\cdot\vec{r}}]e^{+i\omega t}\}$$

$$= -A_0\{\hat{a}\,\vec{e}\cdot[\int d\vec{r}^3 \vec{j} e^{i\vec{k}\cdot\vec{r}}]e^{-i\omega t} + \hat{a}^+\vec{e}\cdot[\int d\vec{r}^3 \vec{j} e^{+i\vec{k}\cdot\vec{r}}]^* e^{+i\omega t}\}$$

W.L.O.G. $\vec{j} = \vec{j}^* \in \mathbb{R}$eal. Let $\int d\vec{r}^3 \, \vec{j}(\vec{r},t) e^{i\vec{k}\cdot\vec{r}} \equiv \vec{J}(\vec{k},t)$

be the 3D Fourier xform of $\vec{j}(\vec{r},t)$ in $\vec{k}$-space.

(Note: $\hbar = c = G = \sqrt{2\pi} = 1$.)

$$\Rightarrow \boxed{\hat{U}(t) = -A_0\{\hat{a}[\vec{e}\cdot\vec{J}]e^{-i\omega t} + \hat{a}^+[\vec{e}\cdot\vec{J}^*]e^{+i\omega t}\}}$$

is the time dependent interaction Hamiltonian.

Recall (Merzbacher) in the Dirac interaction picture from $t \to t+\delta t$ where $\delta t \ll 1$ we may approximate

$$\Delta\hat{U}(t) \approx \hat{U}(t) - \hat{U}(t+\delta t) \approx \left(\frac{\partial\hat{U}(t)}{\partial t}\right)\delta t$$

↓ MOR

$$= +iA_0\omega\{\hat{a}(\vec{e}\cdot\vec{J})e^{-i\omega t} - \hat{a}^+(\vec{e}\cdot\vec{J}^*)e^{+i\omega t}\}\delta t$$

The corresponding differential time evolution op is

$\hat{G}(\delta t) \equiv \exp[-i\hbar \Delta \hat{V} \delta t]$ interaction picture [MERTZ]

Let: $u(t) \equiv -\hbar A_0 (\vec{e} \cdot \vec{J}^*(t)) e^{+i\omega t}$ (Different in our book)

where $\omega \delta t^2 \approx \delta t$ [Richtmayer, Kenard, Cooper]

Then $\boxed{\hat{G}(\delta t) = \exp[\alpha(t)\,\hat{a}^\dagger - \alpha^*(t)\,\hat{a}]} = \hat{D}_{\alpha(t)}$

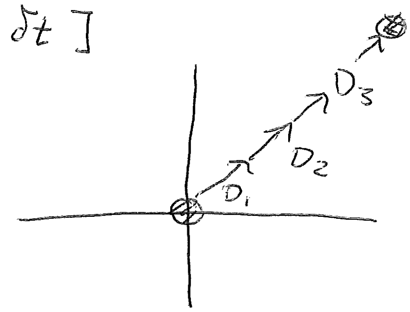
This is THE DISPLACEMENT op $\alpha(t) \equiv u(t)\,\delta t$

So classical current $\vec{J}$ induces coherent state
That's how we make radio waves: $\bar{n} \to \infty$.

For evolution not small $t: 0 \to T$ divide interval into L chunks of δt s.t. $\ell = 0, \ldots, L$ and $L = T/\delta t$ and $t \to t_\ell = \ell\,\delta t$

À LA MERZBACHER:

$$\hat{G}(T) = \lim_{\delta t \to 0} \prod_{\ell=0}^{L} \hat{D}[\overbrace{u(t_\ell)\,\delta t}^{\alpha_\ell}]$$



$$= \lim_{\delta t \to 0} \{\underbrace{\hat{D}_L \times \hat{D}_{L-1} \times \cdots \hat{D}_1 \times \hat{D}_0}_{\text{time ordered}}\}$$

$$= \lim_{\delta t \to 0} e^{i\Phi_L} \hat{D}[\alpha_L + \alpha_{L-1} + \cdots + \alpha_1 + \alpha_0] \quad \text{Eq. 3.42}$$

$$= \lim_{\delta t \to 0} e^{i\Phi_L} \hat{D}[\alpha_L(T)]$$

where Φ_L is some overall phase I'm a gonna drop ...

and $\alpha_L(T) \equiv \sum_{\ell=0}^{L} \alpha_\ell \equiv \sum_{\ell=0}^{L} u(t_\ell)\,\delta t$

which is a Rieman sum definition of an integral

$$\lim_{\delta t \to 0} \alpha_L(t) = \lim_{\delta t \to 0} \sum_{\ell=0}^{T/\delta t} u(t_\ell)\,\delta t \equiv \int_0^T u(t)\,dt \equiv \alpha(T)$$

$$= -\hbar A_0\, \vec{e} \cdot \int_0^T dt\, \vec{J}^*(t)\, e^{i\omega t} = \text{time integrated interaction erg.}$$

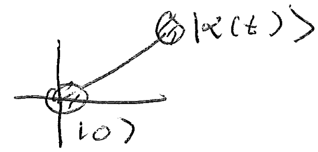
$T \to t$

$\Rightarrow \quad \hat{\mathcal{T}}(t) \equiv \hat{D}_{\alpha(t)}$

and time evolution $\Rightarrow$ generation of coherent state.

$$\boxed{\hat{\mathcal{T}}(t)\,|0\rangle = |\alpha(t)\rangle}$$

Any classical current generates a coherent E & M field



Classical Pump generates laser!

Classical current generates radio waves!

How many photons in a radio wave?

Eq. 3.13 $\Rightarrow \quad \langle\alpha|\hat{\mathcal{E}}|\alpha\rangle = 2\sqrt{\bar{n}}\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} = 1\ \frac{\text{Volt}}{\text{meter}}$

$\uparrow$ E-Field

$= \frac{1\ \text{Volt}}{1\ \text{meter}} \quad \Rightarrow$

Radiowave $\omega \sim 100\ \text{MHz} = 10^{8}\ \text{Hz}$

$\mathcal{E}_0 = \left(\frac{\hbar\omega}{\epsilon_0 V}\right)$ Let $V = 1\,(\text{meter})^3$ = Volume

single photon field

$\downarrow$

$\mathcal{E}_0 \sim \left(\frac{10^{-34} \times 10^{8}}{10^{-11}}\right) \sim 10^{-34} \times 10^{19} \sim 10^{-15}\ \frac{\text{Volt}}{\text{Meter}}$

$\mathcal{E}_{\bar{n}} \sim \sqrt{\bar{n}}\,\mathcal{E}_0 \sim \sqrt{\bar{n}}\,10^{-15}\ \frac{\text{Volt}}{\text{meter}} \sim 1\ \frac{\text{Volt}}{\text{meter}}$

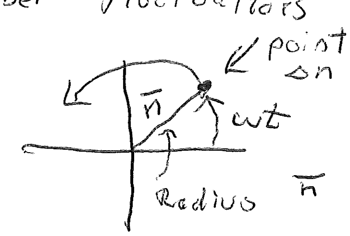
$$\Rightarrow \quad \boxed{\bar{n} \sim 10^{30} \text{ photons in a radio wave}}$$

Them's a lot of photons

$\frac{\Delta n}{\bar{n}} = \frac{\sqrt{\bar{n}}}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} = 10^{-15}$ Number fluctuations

One part in 10^{15} so unobservable

As classical as you can get!



Coherent states are normal $\langle\alpha|\alpha\rangle = 1$ but <u>not</u> orthogonal! $\langle\alpha|\beta\rangle \neq 0$ if $|\alpha\rangle \neq |\beta\rangle$

$$|\alpha\rangle = \sum_n e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\beta\rangle = \sum_m e^{-\frac{|\beta|^2}{2}} \frac{\beta^m}{\sqrt{m!}} |m\rangle$$

$$\langle\beta| = \{|\beta\rangle\}^\dagger = \sum_m e^{-\frac{|\beta|^2}{2}} \frac{\beta^{*m}}{\sqrt{m!}} \langle m|$$

$$\langle\beta|\alpha\rangle = e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} \sum_{mn} \frac{\beta^{*m}\alpha^n}{\sqrt{m!\,n!}} \overbrace{\langle m|n\rangle}^{\delta_m^n}$$

$$= e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} \sum_n \frac{(\beta^*\alpha)^n}{n!} \longleftarrow \underline{\exp}$$

$$= e^{-\left[\frac{|\alpha|^2+|\beta|^2}{2}\right]} e^{\beta^*\alpha}$$

$$= e^{-\frac{1}{2}\left[|\alpha|^2 - 2\beta^*\alpha + |\beta|^2\right]}$$

Noting: $|\beta-\alpha|^2 \equiv (\beta-\alpha)^\dagger(\beta-\alpha) = (\beta^*-\alpha^*)(\beta-\alpha)$

$$= |\beta|^2 - \alpha^*\beta - \beta^*\alpha + |\alpha|^2$$

$$= |\alpha|^2 + |\beta|^2 + \underbrace{2\beta^*\alpha - 2\beta^*\alpha}_{=0} - \alpha^*\beta - \beta^*\alpha \quad \text{complete the square}$$

$$= |\alpha|^2 - 2\beta^*\alpha + |\beta|^2 + \beta^*\alpha - \alpha^*\beta$$

$$\Rightarrow |\alpha|^2 - 2\beta^*\alpha + |\beta|^2 = |\beta-\alpha|^2 + \alpha^*\beta - \beta^*\alpha$$

$$\Rightarrow \boxed{\langle\beta|\alpha\rangle = e^{\frac{1}{2}[\beta^*\alpha - \alpha^*\beta]} e^{-\frac{1}{2}|\beta-\alpha|^2}}$$

$\alpha = \alpha' + i\alpha''$ $\quad \beta = \beta' + i\beta''$ $\quad z = \beta^*\alpha = x + iy$

$\alpha^* = \alpha' - i\alpha''$ $\quad \beta^* = \beta' - i\beta''$ $\quad \Rightarrow z^* = \alpha^*\beta = x - iy$

$\Rightarrow z - z^* = 2iy$

$$\Rightarrow \quad \beta^*\alpha - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha^*\beta)$$

$$\Rightarrow \quad \langle\beta|\alpha\rangle = e^{i\,\mathrm{Im}(\alpha^*\beta)}\, e^{-\frac{1}{2}|\beta-\alpha|^2}$$

$$\langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle = e^{-i\,\mathrm{Im}(\alpha^*\beta)}\, e^{-\frac{1}{2}|\beta-\alpha|^2}$$

$$\Rightarrow \quad \boxed{|\langle\beta|\alpha\rangle|^2 = \langle\beta|\alpha\rangle\langle\alpha|\beta\rangle = e^{-|\beta-\alpha|^2} = e^{-|\alpha-\beta|^2}}$$

This is never zero! let $\alpha = |\alpha| e^{i\theta} = \sqrt{\bar n}\, e^{i\theta}$

$\beta = |\beta| e^{i\varphi} = \sqrt{\bar m}\, e^{i\varphi}$

$$|\alpha-\beta|^2 = \cancel{\quad} (\alpha-\beta)^*(\alpha-\beta)$$

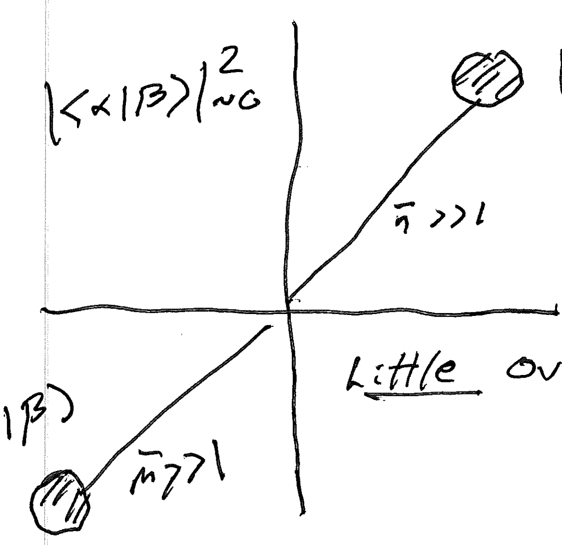
$$= |\alpha|^2 + |\beta|^2 - |\alpha||\beta| e^{i(\theta-\varphi)} - |\alpha||\beta| e^{-i(\theta-\varphi)}$$

$$= \sqrt{\bar n}^{\,2} + \sqrt{\bar m}^{\,2} - 2\sqrt{\bar n \bar n}\cos(\theta-\varphi)$$

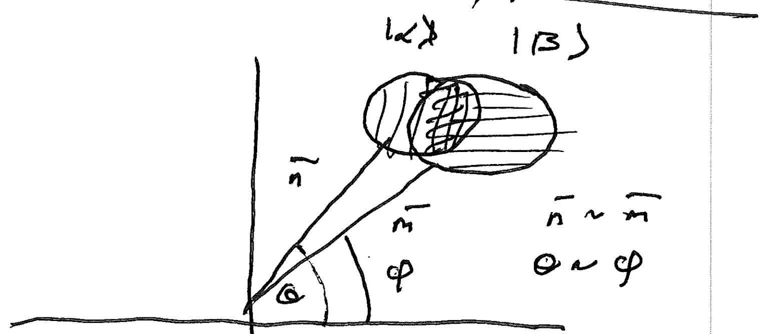
$$\Rightarrow \quad |\langle\beta|\alpha\rangle|^2 = e^{-\bar n}\, e^{-\bar m}\, e^{-2\sqrt{\bar n \bar n}\cos(\theta-\varphi)}$$

This $\to 0$ if $\bar n \to \infty$; $\bar m \to \infty$

and $\theta \neq \varphi$ so classical and <u>well</u> <u>separated</u>



<u>Little</u> overlap



$\Rightarrow |\langle\alpha|\beta\rangle|^2 \sim 1$

<u>Large</u> overlap

$\boxed{\Theta}$ coherent states are over complete

$$\boxed{\frac{1}{\pi}\int d\alpha^2 \;\; |\alpha\rangle\langle\alpha| = \hat{1}}$$

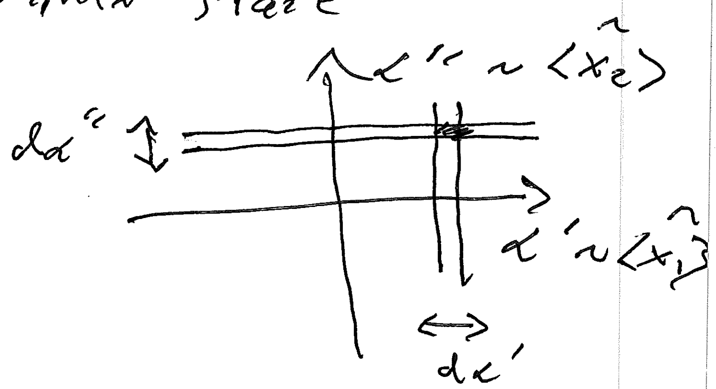
where $d\alpha^* \equiv d\alpha' d\alpha'' = d\,\mathrm{Real}(\alpha)\, d\,\mathrm{Im}(\alpha)$

if $\alpha \in$ Complex $\therefore$ $\alpha = \alpha' + i\alpha''$

with $\alpha', \alpha'' \in$ Real

Double Integral over Complex space

Proof $\left(\bar{n} \equiv |\alpha|^2\right)$



$$|\alpha\rangle = e^{-\frac{\bar{n}}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle\alpha| = e^{-\frac{\bar{n}}{2}} \sum_m \frac{\alpha^{*m}}{\sqrt{m!}} \langle m|$$

Ket - Bra op : $|\alpha\rangle\langle\alpha| = e^{-\bar{n}} \sum_{nm} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!\,n!}} |n\rangle\langle m|$

$$\int d\alpha^2 \, |\alpha\rangle\langle\alpha| = \int d\alpha^2 \sum_{nm} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!\,m!}} |n\rangle\langle m| \, e^{-\bar{n}}$$

Let $\alpha = \alpha' + i\alpha'' = |\alpha| e^{i\theta} = \sqrt{\bar{n}}\, e^{i\theta} \equiv r e^{i\theta}$

$$\int d\alpha^2 \equiv \int d\alpha' \int d\alpha'' = \iint d\alpha' d\alpha'' \rightarrow \int_0^\infty r\,dr \int_0^{2\pi} d\theta$$

switch to polar $r^2 \equiv \bar{n}$

$$\Rightarrow \int d\alpha^2 \, |\alpha\rangle\langle\alpha| = \sum_{nm} \int_0^\infty r\,dr \int_0^{2\pi} d\theta \;\; r^{n+m} e^{-r^2} e^{i(n-m)\theta} \frac{|n\rangle \langle m|}{\sqrt{n!}\,\sqrt{m!}}$$

But $\int_0^{2\pi} d\theta \, \exp[i(n-m)\theta] \equiv 2\pi \delta_n^m$

$$\Rightarrow \int d\alpha^2 |\alpha\rangle\langle\alpha| = 2\pi \sum_n \left(\int_0^\infty dr \; r^{2n+1} e^{-r^2} \right) \frac{|m\rangle\langle n|}{n!}$$

Let $u \equiv r^2$ $\quad du = 2r\,dr \quad$ $r: 0 \rightarrow \infty$

$u: 0 \rightarrow \infty$

$$\Rightarrow \quad \int d\alpha^2 |\alpha\rangle\langle\alpha| = \pi \sum_n \left(\int_0^\infty du \; u^n e^{-u} \right) \frac{|n\rangle\langle n|}{n!}$$

D.I.P. – S.H.I.T.

$$\int_0^\infty u^n e^{-u} = -n! \, e^{-u} \Big|_0^\infty$$

$$\begin{array}{l|l} u^n & e^{-u} \\ \hline n u^{n-1} & \oplus \\ \hline n(n-1) u^{n-2} & \ominus \\ \hline \vdots & \\ \hline & \ominus^{n+1} \\ n! \, u^0 & (-)^n \; e^{-u} \end{array}$$

$$= -n! \, [0 - 1] = n!$$

$$\Rightarrow \quad \int d\alpha^2 |\alpha\rangle\langle\alpha| = \pi \sum_n \frac{n!}{n!} |n\rangle\langle n| = \pi \sum_n |n\rangle\langle n| = \mathbb{1} \cdot \pi$$

$$\Rightarrow \quad \boxed{\frac{1}{\pi} \int d\alpha^2 |\alpha\rangle\langle\alpha| \equiv \hat{\mathbb{1}}}$$