3.3 Time Evolution of $|\alpha\rangle$

Recall $\hat{q} \equiv \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a}+\hat{a}^{+})$ $\qquad \hat{p} \equiv \sqrt{\frac{\hbar\omega}{2}}\,\frac{(\hat{a}-\hat{a}^{+})}{i}$

$$= \sqrt{\frac{2\hbar}{\omega}}\,\hat{X}_1$$

$$= \sqrt{\frac{2\hbar}{\omega}}\,\hat{Q}$$

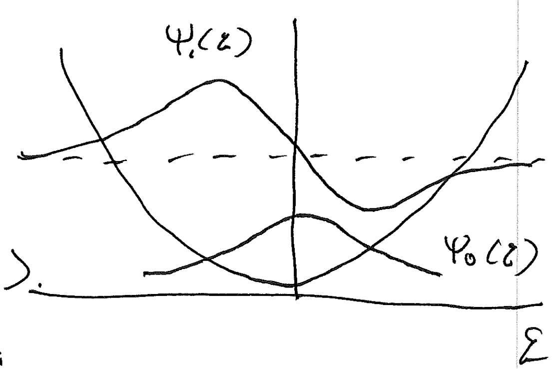
The eigenstates $\hat{q}\,|q\rangle = q\,|q\rangle$ are the "position" states in the coordinate representation.

$|n\rangle$ are the number states in the Fock representation

The probability amplitude that state $|n\rangle$ has position $|q\rangle$ is solution to Schröding Eq. S.H.O.

$$\langle q|n\rangle \equiv \psi_n(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\, e^{-\xi^2/2} H_n(\xi) \qquad *$$

where $\xi = q/q_0$ and $q_0 = \sqrt{\hbar/\omega}$ and H_n are Hermite Polynomials



We want to represent $|\alpha\rangle$ in terms of $|q\rangle$ to get "spatial" dependence for $|\alpha\rangle$.

Let $\left(\frac{\omega}{\pi\hbar}\right)^{1/4} \equiv \frac{1}{\sqrt[4]{\pi}}\sqrt{1/q_0} \equiv A$

$$\psi_\alpha(q) \equiv \langle q|\alpha\rangle = \langle q|\sum_n e^{-\bar{n}/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-\bar{n}/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\langle q|n\rangle \overset{*}{=} A e^{-\bar{n}/2} e^{-\xi^2/2}\sum_n \frac{\alpha^n}{2^{n/2} n!} H_n(\xi)$$

Recall Schaum's Hermite Gen. Fcn:

$$\sum_{n=0}^{\infty} \frac{H_n(\xi)\,t^n}{n!} \equiv e^{2t\xi - t^2}$$ ← expand Exp in series H_n are coeff.

Typo in book?

$$\Rightarrow \quad \psi_\alpha(q) = A\, e^{-\frac{\bar{n}}{2}}\, e^{-\frac{\xi^2}{2}} \sum \frac{H_n(\xi)\, t^n}{n!}$$

where $t^n = (\alpha/\sqrt{2})^n \quad \Rightarrow \quad \boxed{t \equiv \alpha/\sqrt{2}}$

$$\Rightarrow \quad \psi_\alpha(q) = A\, e^{-\frac{(\bar{n}+\xi^2)}{2}}\, e^{2t\xi - t^2}$$

$$= A\, e^{-\frac{\bar{n}}{2}}\, e^{-\frac{\xi^2}{2}}\, e^{2t\xi}\, e^{-t^2}$$

complete the square

$$= A\, e^{-\bar{n}/2}\, e^{-\frac{\xi^2}{2}}\, \overbrace{e^{+\xi^2-\xi^2}}^{1}\, e^{2t\xi}\, e^{-t^2}$$

$$= A\, e^{-\bar{n}/2}\, e^{+\frac{\xi^2}{2}}\, e^{-[\xi^2 - 2t\xi + t^2]}$$

$$= \boxed{A\, e^{-\bar{n}/2}\, e^{\frac{\xi^2}{2}}\, e^{-(\xi-t)^2}}$$

Eq. 3.47

$$\Rightarrow \quad P_\alpha(q) = |\psi_\alpha(q)|^2 = \psi^*\psi$$

$$\Rightarrow = |A|^2 e^{-\bar{n}}\, e^{\xi^2}\, e^{-(\xi-t^*)^2}\, e^{-(\xi-t)^2}$$

$$= |A|^2 e^{-\bar{n}}\, e^{\xi^2}\, e^{-\xi^2 + 2t^*\xi - t^*t^* - \xi^2 + 2t\xi - t^2}$$

$$= |A|^2 e^{-\bar{n}}\, e^{\xi^2}\, e^{-2\xi^2 + 2\xi(t^*+t) - (t^{*2}+t^2)}$$

$$t = \frac{\alpha}{\sqrt{2}} = \frac{|\alpha|}{\sqrt{2}} e^{i\theta} \quad \Rightarrow \quad t^* + t = 2\frac{|\alpha|}{\sqrt{2}}\, \frac{e^{-i\theta} + e^{i\theta}}{2}$$

$$= \sqrt{2}\,|\alpha| \cos\theta$$

$$= \sqrt{2}\sqrt{\bar{n}} \cos\theta$$

$$t^{*2} + t^2 = \frac{|\alpha|^2}{2}\left(e^{-i2\theta} + e^{2i\theta}\right)$$

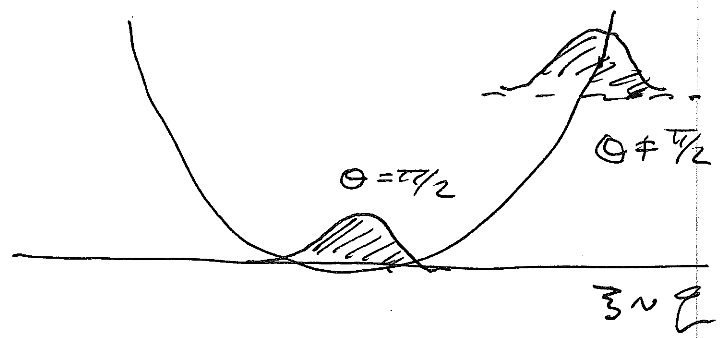
$$= |\alpha|^2 \cos 2\theta$$

$$= |\alpha|^2 [2\cos^2\theta - 1]$$

$$P_\alpha(q) = |A|^2 e^{-\bar{n}+\xi^2}\, e^{-2\xi^2 + 2\xi\sqrt{2}\sqrt{\bar{n}}\cos\theta - \bar{n}[2\cos^2\theta - 1]}$$

(WLOG $\theta = \pi/2$ Initial phase) $= |A|^2 e^{\cancel{-\bar{n}}+\xi^2}\, e^{-2\xi^2 \cancel{+\bar{n}}}$

$$P_\alpha(q) = |A|^2 e^{-\xi^2}$$



a simple Gaussian

check $\int_{-\infty}^{\infty} \psi^* \psi \, dx$

$$= q_0 \int_{-\infty}^{\infty} \psi^* \psi \, d\xi = |A|^2 q_0 \int_{-\infty}^{\infty} e^{-\xi^2} \overset{\text{Schaum's}}{=} |A|^2 q_0 \sqrt{\pi}$$

$= 1$ ✓ $\quad \Theta = \pi/2 \Rightarrow$ Ground state

In general $\quad P_\alpha(q) = |A|^2 e^{-\bar{n}+\xi^2} \exp\left[-(2\xi^2 - 2\sqrt{2}\sqrt{\bar{n}}\cos\Theta\,\xi + \bar{n}\,2\cos^2\Theta - \bar{n})\right]$

Note: $2\xi^2 - \sqrt{2}\,2\sqrt{\bar{n}}\cos\Theta\,\xi + 2\bar{n}\cos^2\Theta - \bar{n}$

$$= 2[\xi^2 - \sqrt{2\bar{n}}\cos\Theta\,\xi + \bar{n}\cos^2\Theta - \bar{n}/2]$$

$\sin^2 + \cos^2 = 1$
$\cos^2 - 1 = -\sin^2$
$\sin^2 - 1 = -\cos^2$

$$= 2[\xi - \sqrt{2\bar{n}}\cos\Theta\,\xi + \bar{n}\cos^2\Theta - \bar{n}/2]$$

$$= 2\left[\xi - \sqrt{2\bar{n}}\cos\Theta\,\xi + \frac{\bar{n}}{2}\cos^2\Theta + \frac{\bar{n}}{2}\cos^2\Theta - \bar{n}/2\right]$$

$$= 2\left[\left(\xi - \sqrt{\frac{\bar{n}}{2}}\cos\Theta\right)^2 - \frac{\bar{n}}{2}\sin^2\Theta\right]$$

$$P_\alpha(q) = |A|^2 e^{-\bar{n}+\xi^2} \exp\left[-2\left(\xi - \sqrt{\frac{\bar{n}}{2}}\cos\Theta\right)^2 + \bar{n}\sin^2\Theta\right]$$

$$= \boxed{|A|^2 e^{-\bar{n}\cos^2\Theta} e^{\xi^2} e^{-2\left(\xi - \sqrt{\frac{\bar{n}}{2}}\cos\Theta\right)^2}}$$

A Gaussian centered at $\boxed{\xi = \bar{\xi} = \sqrt{\frac{\bar{n}}{2}}\cos\Theta}$

OK for time evolution we just got

$\Theta = \Theta(t)$ evolves in time.

If $\hat{a}(0)$ is time independent operator at $t=0$

$\Rightarrow$ $\hat{a}(0)|\alpha\rangle = \alpha|\alpha\rangle$ where

α is time-independent coherent state. WLOG

choose $\theta = 0$ as the initial phase so $\alpha = |\alpha| e^{i\theta} = |\alpha|$

Let $|\alpha(t)\rangle$ solve the time-dependent Schrödinger Eq

$$i\hbar \frac{d\psi(t)}{dt} = \hat{H}\psi(t) \Rightarrow i\hbar \frac{d|\alpha(t)\rangle}{dt} = \hat{H}|\alpha(t)\rangle$$

since $\hat{H} = \hbar\omega(\hat{n}+1/2)$ is independent of t we can integrate [Merzbacher]:

$$|\alpha(t)\rangle = e^{-i\frac{1}{\hbar}\hat{H}t}|\alpha(0)\rangle = e^{-i\omega t(\hat{n}+1/2)}|\alpha(0)\rangle$$

$$= e^{-i\omega t/2}\, e^{-i\omega t\hat{n}}\,|\alpha(0)\rangle = e^{-i\omega t/2}\left[\sum_m \frac{(-i\omega t\hat{n})^m}{m!}\right] \times \left[\sum_n e^{-\bar{n}/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right]$$

$$= e^{-i\omega t/2}\sum_{mn} e^{-\bar{n}/2}\frac{[-i\omega t]^m}{m!}\frac{\alpha^n}{\sqrt{n!}}\hat{n}^m|n\rangle$$

$$= e^{-i\omega t/2}\sum_{mn} e^{-\bar{n}/2}\frac{[-i\omega t]^m}{m!}\frac{\alpha^n}{\sqrt{n!}} n^m|n\rangle$$

$$= e^{-i\omega t/2}\sum_n e^{-\bar{n}/2}\, e^{-i\omega t n}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \text{EQ 3.49}$$

$$= e^{-i\omega t/2}\sum_n e^{-\bar{n}/2}\frac{(e^{-i\omega t}\alpha)^n}{\sqrt{n!}}|n\rangle \equiv \boxed{e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle}$$

Again $\alpha = |\alpha|$ means $\alpha(t) = \alpha_0 e^{-i\omega t}$

OK Let $\theta \rightarrow \theta(t) = -\omega t$

$$P_{\alpha(t)}(q) \equiv |\langle q|\alpha(t)\rangle|^2 = \langle q|\alpha(t)\rangle^* \langle q|\alpha(t)\rangle$$

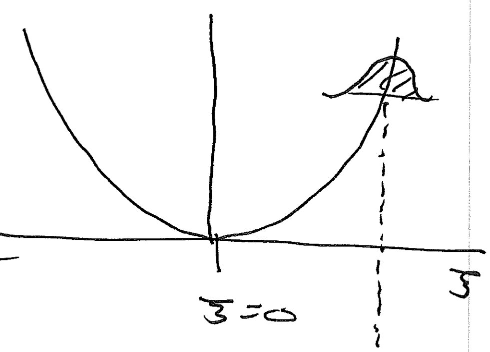
$$= [\cancel{e^{-i\omega t/2}} \langle q|\alpha e^{-i\omega t}\rangle]^* [\cancel{e^{-i\omega t/2}} \langle q|\alpha e^{-i\omega t}\rangle]$$

$$= |\langle q|\alpha e^{-i\omega t}\rangle|^2 = |\langle q|\alpha e^{i\theta(t)}\rangle|^2$$

where $\alpha \in$ Real $|\alpha| = \alpha$. Plug $\theta(t)$ into previous formula! $\theta \rightarrow -\omega t$

$$P_{\alpha(t)}(q) = |A|^2 e^{-\bar{n}\cos^2(\omega t)} e^{\xi^2} e^{-2(\xi - \sqrt{\frac{\bar{n}}{2}}\cos\omega t)^2}$$

The position of the gaussian oscillates about $\xi = 0$ with frequency $\omega \Rightarrow$ classical



Note $\xi = q/q_0$

$$\Rightarrow e^{-2(\xi - \sqrt{\frac{\bar{n}}{2}}\cos\omega t)^2} = e^{-\frac{(q - q_0\sqrt{\frac{\bar{n}}{2}}\cos\omega t)^2}{1/2 \cdot q_0}}$$

so gaussian has width $2\sigma = \frac{1}{2} q_0 \Rightarrow$

$$\Delta q = \sigma = \frac{1}{4} q_0 = \sqrt{\frac{\hbar}{8\omega}}$$

The amplitude of oscillation $= q_{MAX} = q_0\sqrt{\frac{\bar{n}}{2}}$