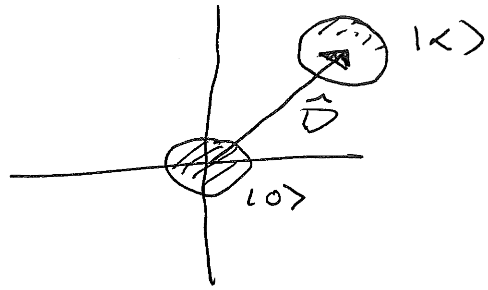
3.2 Displacement $\hat{op}$

A coherent state is a displaced $|0\rangle$ state: what is Displacement $\hat{op}$ s.T.



$$\boxed{|\alpha\rangle \equiv \hat{D}_\alpha |0\rangle}$$

<u>Motivation:</u> $\quad |\alpha\rangle = \hat{D}_\alpha |0\rangle$

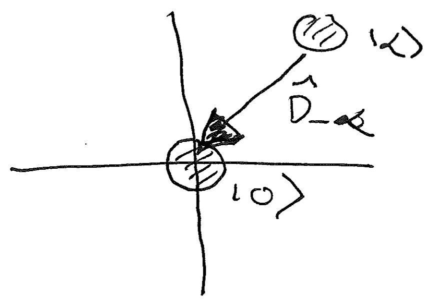
$\Rightarrow \quad e^{-\bar{n}/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \hat{D}_\alpha |0\rangle \quad$ but $\quad |n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}} |0\rangle$

$\Rightarrow \quad e^{-\bar{n}/2} \sum_n \frac{(\alpha \hat{a}^+)^n}{n!} |0\rangle = \hat{D}_\alpha |0\rangle$

$$\overset{?}{\Rightarrow} \quad \boxed{\hat{D}_\alpha = e^{-\bar{n}/2} \sum_n \frac{(\alpha \hat{a}^+)^n}{n!} \equiv e^{-\bar{n}/2} e^{\alpha \hat{a}^+}}$$

<u>close!</u> but we want

$\hat{D}_{-\alpha} |\alpha\rangle = |0\rangle \quad$ <u>Inverse</u>



or $\quad \hat{D}_\alpha \hat{D}_{-\alpha} = \mathbb{1}$

even better $\quad \hat{D}_{-\alpha} \equiv \hat{D}^+ \quad \Rightarrow \quad \hat{D}^+ \hat{D} = \hat{D} \hat{D}^+ = \mathbb{1}$

and $\hat{D}_\alpha$ is <u>unitary!</u> $\quad$ Note $\hat{\Theta}^+ = -\hat{\Theta}$

$\Rightarrow \quad e^{\hat{\Theta}}$ is unitary $\quad [e^{\hat{\Theta}}]^+ [e^{\hat{\Theta}}] = e^{-\hat{\Theta}} e^{\hat{\Theta}} = \mathbb{1}$

since $\hat{\Theta} = \alpha \hat{a}^+$ does not satisfy $\hat{\Theta}^+ = -\hat{\Theta}$

we need a new exponent, $\quad$ since $(\alpha \hat{a}^+)^+ = \alpha^* \hat{a}$

Let $\quad \hat{\Theta} \equiv \alpha \hat{a}^+ - \alpha^* \hat{a}$

$\Rightarrow \quad \hat{\Theta}^+ = -[\alpha \hat{a}^+ - \alpha^* \hat{a}] = -\hat{\Theta}$

$\Rightarrow \quad \exp[\alpha \hat{a}^+ - \alpha^* \hat{a}]$ is <u>unitary!</u>

check

C.B.H. $\quad e^{\hat{X}+\hat{Y}} = e^{\hat{X}} e^{\hat{Y}} e^{-\frac{[\hat{X},\hat{Y}]}{2}}$ *

$$\equiv e^{\hat{Y}} e^{\hat{X}} e^{+\frac{[\hat{X},\hat{Y}]}{2}} \quad **$$

$\hat{X} = \alpha \hat{a}^{+} \qquad \hat{Y} = -\alpha^{*}\hat{a} \qquad e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}} = e^{\hat{X}+\hat{Y}}$

$$[\hat{X},\hat{Y}] = -|\alpha|^2 [\hat{a}^{+}, \hat{a}] = +|\alpha|^2 [\hat{a}, \hat{a}^{+}]$$

$$= \bar{n} \cdot \hat{1} = \bar{n}$$

$$\Rightarrow \quad e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}} = e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}} e^{-\bar{n}/2}$$

But $\quad e^{-\bar{n}/2} e^{\alpha\hat{a}^{+}} e^{-\alpha^{*}\hat{a}} |0\rangle$

$$= e^{-\bar{n}/2} e^{\alpha\hat{a}^{+}} \underbrace{\sum_n \frac{(-\alpha^{*}\hat{a})^n}{n!} |0\rangle}_{1\cdot|0\rangle + 0|0\rangle + 0|0\rangle + \cdots}$$

$$= e^{-\bar{n}/2} e^{\alpha\hat{a}^{+}} |0\rangle = e^{-\bar{n}/2} \sum_n \frac{(\alpha\hat{a}^{+})^n}{n!} |0\rangle$$

$$= e^{-\bar{n}/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \equiv |\alpha\rangle \quad ☺$$

OK Let us Define $\boxed{\hat{D}_\alpha \equiv e^{\alpha\hat{a}^{+} - \alpha^{*}\hat{a}}}$

By construction $\quad \hat{D}_\alpha |0\rangle = |\alpha\rangle$ ✓

$$\hat{D}_\alpha^{+} = \hat{D}_{-\alpha} \quad ✓$$

$$\hat{D}_\alpha^{+} \hat{D}_\alpha = \hat{D}_{-\alpha} \hat{D}_\alpha = \mathbb{1} \quad ✓$$

Note $\quad ** \Rightarrow \quad \hat{D}_\alpha = e^{\frac{\hbar}{2}} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^+}$

$* \Rightarrow \quad \hat{D}_\alpha = e^{-\frac{\hbar}{2}} e^{\alpha \hat{a}^+} e^{-\alpha^* \hat{a}}$

$$* \Rightarrow \quad \hat{D}_\alpha^+ = e^{-\frac{\hbar}{2}} \left[e^{-\alpha^* \hat{a}} \right]^+ \left[e^{\alpha \hat{a}^+} \right]^+$$

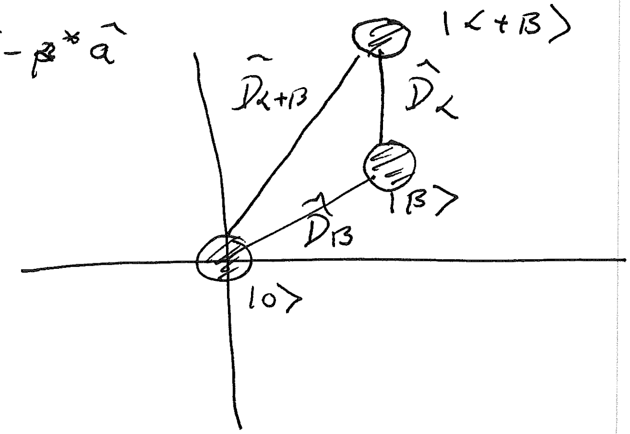
$$= e^{-\frac{\hbar}{2}} e^{-\alpha \hat{a}^+} e^{\alpha^* \hat{a}}$$

$$= e^{-\hbar/2} e^{(-\alpha) \hat{a}^+} e^{-(-\alpha)^* \hat{a}} \equiv \hat{D}_{-\alpha}$$

$** \Rightarrow \quad \hat{D}_\alpha^+ = \hat{D}_{-\alpha} \quad \checkmark$

Lemma $\quad \boxed{\hat{D}_\alpha \hat{D}_\beta = e^{i \, Im \{\alpha \beta^*\}} \hat{D}_{\alpha+\beta}}$

$$\hat{D}_\alpha \hat{D}_\beta \equiv e^{\alpha \hat{a}^+ - \alpha^* \hat{a}} \cdot e^{\beta \hat{a}^+ - \beta^* \hat{a}}$$



$\hat{X} \equiv \alpha \hat{a}^+ - \alpha^* \hat{a}$

$\hat{Y} \equiv \beta \hat{a}^+ - \beta^* \hat{a}$

$$[\hat{X}, \hat{Y}] = \alpha\beta \underbrace{\cancel{[\hat{a}^+, \hat{a}^+]}}_{\to 0}$$

$$- \alpha\beta^* [\hat{a}^+, \hat{a}]$$

$$- \alpha^*\beta [\hat{a}, \hat{a}^+] + \alpha^*\beta^* \underbrace{\cancel{[\hat{a}, \hat{a}]}}_{\to 0}$$

$$= +\alpha\beta^* - \alpha^*\beta = + \left[(\alpha^*\beta)^* - (\alpha^*\beta) \right]$$

$z = u + iv$ $\qquad z + z^* = 2u = 2 Re\, z$

$z^* = u - iv$ $\qquad z - z^* = 2iv = 2i\, Im\, z$ $\Big\} \Longrightarrow 2i\, Im \{\alpha^* \beta\}$

$\equiv 2i\, Im \{\alpha \beta^*\}$

CBH $\quad e^{\hat{X}} e^{\hat{Y}} = e^{\frac{[\hat{X}, \hat{Y}]}{2}} e^{\hat{X} + \hat{Y}} =$

$$= e^{i\, Im \{\alpha\beta^*\}} e^{(\alpha+\beta) \hat{a}^+ - (\alpha+\beta)^* \hat{a}}$$

$$\equiv e^{i\, Im \{\alpha\beta^*\}} D_{\alpha+\beta} \quad \therefore$$

When Life Gives You Lemmas? Make Lemmanade!

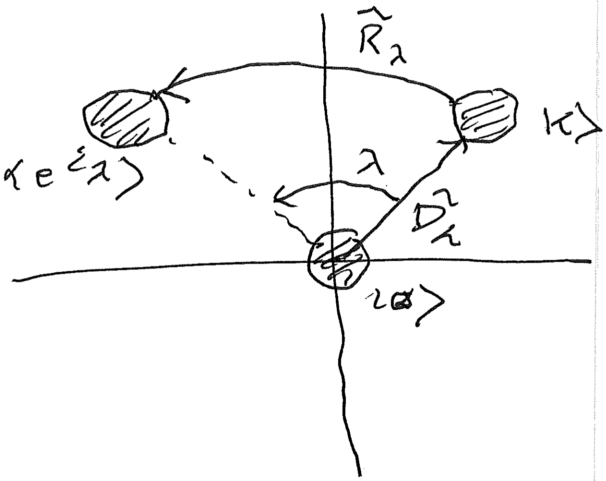
$\forall\ \lambda \in$ Real and $|\alpha\rangle \in \mathcal{H}_{SHO}$ coherent states

$\exists$ a rotation operator $\hat{R}_\lambda \equiv e^{i\lambda\hat{n}} \equiv \sum_m \frac{(i\lambda\hat{n})^m}{m!}$

s.t.

$$\boxed{\hat{R}_\lambda |\alpha\rangle = |\alpha e^{i\lambda}\rangle}$$

where $|\alpha e^{i\lambda}\rangle$ is $|\alpha\rangle$ rotated by $\measuredangle \lambda$ in phase space.



Proof

$\hat{R}_\lambda |\alpha\rangle$

$$\equiv \sum_m \frac{(i\lambda\hat{n})^m}{m!} |\alpha\rangle$$

$$\equiv \sum_{mn} \frac{(i\lambda\hat{n})^m}{m!} e^{-\bar{n}/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \sum_{mn} \frac{(i\lambda n)^m}{m!} e^{-\bar{n}/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \sum_n e^{i\lambda n} e^{-\bar{n}/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \sum_n e^{-\bar{n}/2} \frac{(e^{i\lambda}\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$\equiv |\alpha e^{i\lambda}\rangle \quad \therefore$$