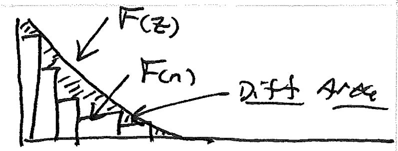
2.6 The Casimir Effect 1



We'll need the Euler-Maclaurin summation formula Eq. 21.45 Schaum's Outline / Math Handbook

$$\sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dz\, F(z) = \sum_{p=1}^{\infty} (-1)^{p-1} \frac{B_p}{2p!} \left\{ F^{(2p-1)}(\infty) - F^{(2p-1)}(0) \right\}$$

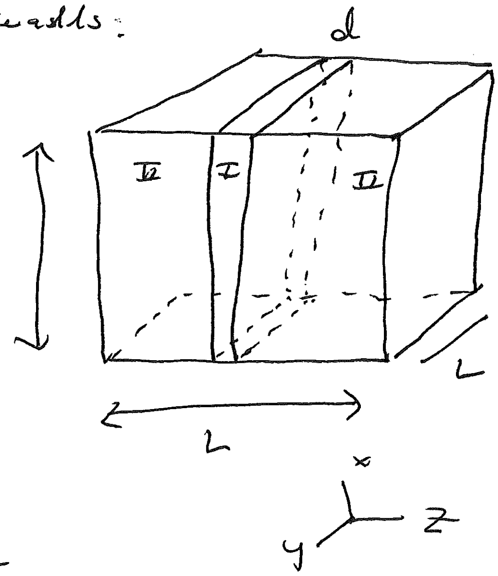
where $B_p \in \{ 1/6, 1/30, 1/42, \dots \}$ are Bernoulli #s (Schaum's Page 139) and $F^{(m)}(z) \equiv \frac{d^m}{dz^m} F(z)$

This is a formula for estimating the difference between the sum $\sum F(n)$ and the integral $\int F(z)\,dz$

Consider two boxes with conducting walls:

I: $L_x = L_y = L \to \infty$, $L_z = d$

II $L_x = L_y = L_z = L \to \infty$



The walls in close B.C. $\Rightarrow$

$$\omega^{I}_{lmn} = \pi c \left[\frac{l^2}{L^2} + \frac{m^2}{L^2} + \frac{n^2}{d^2} \right]^{1/2}$$

Out far

$$\omega^{II}_{lmn} = \pi c \left[\frac{l^2}{L^2} + \frac{m^2}{L^2} + \frac{n^2}{L^2} \right]^{1/2}$$

The energy in both boxes is infinite. The energy difference $E_{II} - E_{I}$ will turn out to be finite and is energy between two large conducting plates - in empty space - at $T = 0$.

$$E_I = \sum_{nlm}{}' \underbrace{(2)}_{\text{polarization}} \tfrac{1}{2} \underbrace{\hbar \omega^{I}_{lmn}}_{\text{vacuum erg}}$$

The "′" means that no "(2)" if $l=0$, or $m=0$, or $n=0$ D.C. no polarization.

$$E_{II} = \sum_{nlm}{}' (2) \frac{1}{2} \hbar \omega^{II}_{lmn}$$

As $L \to \infty$ k_x and k_y become continuous in I

k_x, k_y, and k_z become continuous in II

$k_x = \frac{l\pi}{L}$ $\quad k_y = \frac{m\pi}{L}$ $\quad k_z = \frac{n\pi}{d}$

$dk_x = \frac{\pi}{L}dl$ $\quad dk_y = \frac{\pi}{L}dm$ $\quad dk_z = \frac{\pi}{d}dn$

$\sum_l \equiv \sum_l dl \to \int \frac{L}{\pi} dk_x$ Dimensionless = Dimensionless

$\sum_m \equiv \sum_m dm \to \int \frac{L}{\pi} dk_y$

$\sum_n \equiv \sum_n dn \to \int \frac{d}{\pi} dk_z$

$$E^{I} \to \hbar c \left(\frac{L}{\pi}\right)^2 \sum_n \int dk_x \int dk_y \left[k_x^2 + k_y^2 + \frac{n^2\pi^2}{d^2}\right]^{1/2}$$

$$E^{II} \to \hbar c \left(\frac{L}{\pi}\right)^2 \left(\frac{d}{\pi}\right) \iiint dk_x\, dk_y\, dk_z \left[k_x^2 + k_y^2 + k_z^2\right]^{1/2}$$

Convert to cylindrical coordinates $\quad k_x^2 + k_y^2 = k^2$

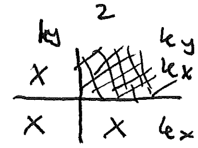
$k_z = k_z$

$dk_x\, dk_y = k\, dk\, d\varphi$

$$E^{I} = \frac{\pi}{2}\hbar c \left(\frac{L}{\pi}\right)^2 \sum_n \int dk\, k \left[k^2 + \frac{n^2\pi^2}{d^2}\right]^{1/2}$$

$\int_0^{\pi/2} d\varphi = \frac{\pi}{2}$

$$E^{II} = \frac{\pi}{2}\hbar c \left(\frac{L}{\pi}\right)^2 \left(\frac{d}{\pi}\right) \int dk\, k \int dk_z \left[k^2 + k_z^2\right]^{1/2}$$



$$U = E^{I} - E^{II} = \frac{\pi}{2}\hbar c \left(\frac{L}{\pi}\right)^2 \left\{ \sum_n \int dk\, k \left(k^2 + \frac{n^2\pi^2}{d^2}\right)^{1/2} - \frac{d}{\pi}\int dk_z \int dk\, k \left(k^2 + k_z^2\right)^{1/2} \right\}$$

Let $w \equiv \frac{k^2 d^2}{\pi^2}$ $\quad k: 0 \to \infty$ $\quad z^2 = \frac{k_z^2 d^2}{\pi^2}$ $\quad dz = \left(\frac{d}{\pi}\right) dk_z$

$dw = 2k\, dk \left(\frac{d^2}{\pi^2}\right)$ $\quad w \to : 0 \to \infty$ $\quad k\, dk = \frac{1\pi^2}{2d^2}$ $\quad \frac{\pi}{d} dz = dk_z$

$$U = \left(\frac{\hbar c L^2}{4\pi}\right)\left(\frac{\pi}{d}\right)^{3} \left\{ \sum_n{}' \int dw\, (w + n^2)^{1/2} - \int dz \int dw\, (w + z^2)^{1/2} \right\}$$

$$= \frac{\hbar c L^2}{4} \frac{\pi^2}{d^3} \left\{ \sum_n{}' \int dw\, (w + n^2)^{1/2} - \int dz \int dw\, (w + z^2)^{1/2} \right\}$$

Recall $\sum_n{}' = \frac{1}{2}F(0) + F(1) + \cdots = -\frac{1}{2}F(0) + F(0) + \cdots$

$$\Rightarrow \sum_{n=0}^{\infty\prime} F(n) = -\frac{1}{2}F(0) + \sum_{n=0}^{\infty} F(n) = \frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n)$$

Let $F(u) := \int_0^\infty dw\,(w+u^2)^{1/2}$ $\quad u = n, z$

Assume a cut off $F(u) \to F(u)\,e^{-\lambda u}$ s.t. $F(\infty) = F'(\infty) = \cdots = 0$

and then let $\lambda \to 0$ Euler-Mac:

$$\Rightarrow U = \frac{\hbar c L^2}{4}\frac{\pi^2}{d^3}\left\{ \cancel{\frac{1}{2}F(0)} + \sum_{n=1}^{\infty} F(n) - \int dz\, F(z) \right\}$$

Eq 21.45 $\quad \sum_{n=1}^{\infty} F(n) - \int_0^\infty dz\, F(z) \equiv -\cancel{\frac{1}{2}F(0)} - \frac{1}{12}F'(0) + \frac{1}{720}F'''(0) + \cdots$

$$\Rightarrow U = \frac{\hbar c L^2}{4}\frac{\pi^2}{d^3}\left\{ -\frac{1}{12}F'(0) + \frac{1}{720}F'''(0) + \cdots \right\}$$

Let $F(u) = \int_0^\infty dw\,(w+u^2)^{1/2}$

$x = w + u^2$
$dx = dw$
$x: u^2 \to \infty$
Λ = large cutoff

$$\Rightarrow \quad = \int_{u^2}^{\Lambda} dx\,\sqrt{x}$$

$$= \frac{2}{3}x^{3/2}\Big|_{u^2}^{\Lambda} = \frac{2}{3}\Lambda^{3/2} - \frac{2}{3}u^3$$

$\Rightarrow F'(u) = -2u^2 \quad \Rightarrow F'(0) = 0$

$\Rightarrow F''(u) = -4u$

$\Rightarrow F'''(u) = -4 \quad \Rightarrow F'''(0) = -4$

$$\Rightarrow U(d) = -\hbar c\pi^2\frac{L^2}{d^3}\frac{1}{720} = \boxed{-\frac{\pi^2}{720}\hbar c\frac{L^2}{d^3}}$$

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$$\Rightarrow \exists \text{ Force} = F(d) \equiv -\frac{dU}{dd} = -\frac{3\pi^2}{720}\frac{\hbar c}{d^4}L^2$$

$$\Rightarrow \text{Pressure} = \frac{F(d)}{L^2} = \boxed{-\frac{\pi^2}{240}\frac{\hbar c}{d^4}}$$

Two plates in empty space at $T=0$ attract!!!