Geometric series for Ch 2.5

Theorem Θ I: If $|r|<1$ Then $\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$

Proof: If $|r|<1$ series is absolutely convergent and we may manipulate it term by term

Let $S_0 \equiv \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$

$\Rightarrow \quad r S_0 = \sum_{n=0}^{\infty} r^{n+1} = r + r^2 + \cdots$

$\Rightarrow \quad S_0 - r S_0 = 1 \quad \Rightarrow \quad S_0 = \frac{1}{1-r} \quad \therefore$

Corollary Γ I: If $0 \le r < 1$ $\quad P_n \equiv (1-r) r^n$

is a normalized probability distribution.

Proof $0 \le r < 1 \Rightarrow 0 \le r^n < 1$ and $0 \le 1-r < 1 \Rightarrow 0 \le P_n < 1$

Further $\sum_{n=0}^{\infty} P_n = (1-r)\sum_n r^n = \frac{(1-r)}{(1-r)} = 1 \quad \therefore$

Θ II: $|r|<1 \Rightarrow \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2} \equiv S_1$

Proof $\frac{dS_0}{dr} = \frac{d}{dr}\sum_n r^n = \sum_n n r^{n-1} \equiv \frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{1}{(1-r)^2}$

$\Rightarrow \quad r\frac{dS_0}{dr} = \sum_n n r^n \equiv S_1 = \frac{r}{(1-r)^2} \quad \therefore$

Γ II: If $0 \le r < 1$ Then $\langle n \rangle = \bar{n} \equiv \sum_{n=0}^{\infty} n P_n = \frac{r}{(1-r)}$

Θ III: $|r|<1 \Rightarrow \sum_{n=0}^{\infty} n^2 r^n \equiv S_2 = \frac{r(1+r)}{(1-r)^3}$

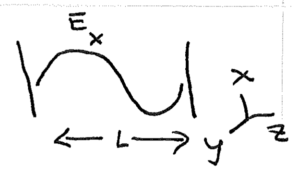
Proof $\frac{d}{dr}\sum_n n r^n = \sum_n n^2 r^{n-1} \equiv \frac{d}{dr}\left[\frac{r}{(1-r)^2}\right] = \cdots = \frac{1+r}{(1-r)^3} \quad \therefore$

$\Rightarrow \quad S_2 = \frac{r(1+r)}{(1-r)^3}$

Γ III: $0 \le r < 1 \Rightarrow \langle n^2 \rangle \equiv \overline{n^2} \equiv \sum_{n=0}^{\infty} n^2 P_n = \frac{r(1+r)}{(1-r)^2}$

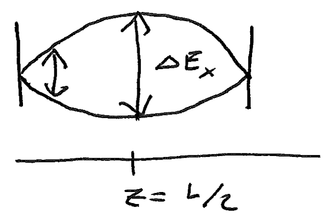
Proof: use Θ III, multiply by $(1-r)$ and note $P_n = (1-r)r^n$

For single mode in a cavity recall electric field that $\Delta E_x^{(n)} = \sqrt{2}\, \mathcal{E}_0 (\sin kz)\sqrt{n+1/2}$



For n photons and $\mathcal{E}_0 = (\hbar\omega/\varepsilon_0 V)^{1/2}$ is the single-photon Electric Field amplitude. Hence the electric field fluctuates vacuum $n=0$ $\Delta E_x^{(0)} \equiv \mathcal{E}_0 \sin kz$

For a cavity $L = \lambda/2 \Rightarrow k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$ and $\sin(\frac{\pi}{L}z)$ is maximum at $z = L/2$



The zero point energy per mode $\vec{k}, s$ is $E_{\vec{k}s} = \frac{\hbar}{2}\omega_k$

In empty space summing over all modes:

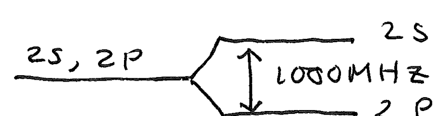
$$E_{ZPE} = \sum_{\vec{k}s} E_{\vec{k}s} = \frac{\hbar}{2}\sum_{\vec{k}s}\omega_k \rightarrow \frac{\hbar}{2}\int \overset{\omega_k}{\downarrow}\!\!\rho(\omega)\,d\omega$$

$$= \frac{\hbar}{2}\frac{V}{\pi^2 c^3}\int_0^\infty \omega^3 d\omega = \frac{\hbar V}{8\pi^2 c^3}\lim_{\omega\to\infty}\omega^4 = \infty$$ Diverges Quite Fast!

Nevertheless ZPE can be viewed as causing real effects such as Lamb shift and Casimir effect.

Lamb shift

In Dirac Eq. for hydrogen the 2S and 2P excited states are degenerate.



But interaction with quantized E & M Field lifts the degeneracy, splitting the levels. Observed by Lamb in 1947, estimated in theory by Bethe 1948. Also in 1948 Welton provided ZPE interpretation.

$E_{ZPE}(\omega) = \frac{1}{2}\hbar\omega$ per mode

but in free space # modes from $\omega \to \omega + d\omega$ is DOM

$$\text{DOM} = V\rho(\omega)\,d\omega = \frac{V\omega^2}{\pi^2 c^3}\,d\omega$$ hence over this range

$$E_{ZPE} = \text{DOM} * \tfrac{1}{2}\hbar\omega = \frac{V\hbar}{2\pi^2 c^3}\,\omega^3 d\omega$$

Now atom only couples to electric field energy

$$E_{ZPE} = \frac{1}{2}\int_V dV\left(\varepsilon_0 \mathcal{E}^2 + \cancel{\frac{1}{\mu_0}B^2}\right) = \frac{\varepsilon_0 V}{2}\,\mathcal{E}^2(\omega)$$

neglect

where $\mathcal{E}$ is electric field at frequency ω.

$$\Rightarrow \quad \frac{\cancel{V}\hbar}{2\pi^2 c^3}\,\omega^3 d\omega = \frac{\varepsilon_0 \cancel{V}}{2}\,\mathcal{E}^2(\omega)$$

$$\Rightarrow \quad \boxed{\mathcal{E}^2(\omega) = \frac{\hbar\omega^3 d\omega}{\pi^2 c^3 \varepsilon_0}}$$

we're going to do this in SI units unlike book

So we assume atom is bathed in a field of this strength.

We also assume the Coulomb potential $V(r) = \frac{-1}{4\pi\varepsilon_0}\frac{e^2}{r}$ is modified by the vacuum field. To estimate how much:

$$\Delta V(x,y,z) = \Delta x\frac{\partial V}{\partial x} + \Delta y\frac{\partial V}{\partial y} + \Delta z\frac{\partial V}{\partial z}$$
$$+ \frac{1}{2}\Delta x^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\Delta y^2\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}\frac{\partial^2 V}{\partial z^2} + \cdots$$

which is a 3D Taylor's series expansion.

OK recall $\langle \mathcal{E} \rangle = 0$ for the vaccum electric field

and only $\Delta \mathcal{E}^2 \neq 0$ (see Eq. 2.44 and Eq. 2.45).

Since for an electron $\Delta x, \Delta y, \Delta y \propto \langle \mathcal{E} \rangle = 0$ only second term survives. If isotropic free space

$\Delta x^2 = \Delta y^2 = \Delta z^2 = \Delta r^2/3$ and

$$\Delta V = \frac{1}{2}\cdot\frac{1}{3}\Delta r^2\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right] = \boxed{\frac{1}{6}\Delta r^2\,\nabla^2 V}$$

If $V(r) = \frac{-1}{4\pi\varepsilon_0} \frac{e^2}{r}$ use NRL Eq. 25 & 24

$$\vec{\nabla} \cdot \vec{\nabla} (1/r) = \vec{\nabla} \cdot \left[-\frac{\vec{r}}{r^3} \right] = -4\pi \delta(\vec{r})$$

$$\Rightarrow \quad \Delta V = \frac{1}{6} \Delta r^2 \left[\frac{4\pi e^2}{4\pi\varepsilon_0} \right] \delta(r) = \frac{1}{6} \frac{e^2}{\varepsilon_0} \Delta r^2 \delta(\vec{r})$$

For a hydrogen state $|n\ell m\rangle = \psi_{n\ell m}(\vec{r})$

the wavefunction vanishes at $\vec{r} = 0$ unless $\ell = m = 0$ (s-state).

Hence the energy shift will be

$$\Delta E = \langle n\ell m | \Delta V | n\ell m \rangle$$

$$= \frac{1}{6} \frac{e^2}{\varepsilon_0} \Delta r^2 \int dV \, \psi^*_{n\ell m}(\vec{r}) \, \delta(\vec{r}) \, \psi_{n\ell m}(\vec{r})$$

$$= \frac{1}{6} \frac{e^2}{\varepsilon_0} \Delta r^2 \, |\psi_{n00}(0)|^2$$

One can prove $|\psi_{n00}(0)|^2 = \frac{1}{\pi n^3 a_0^3}$

where $a_0 = \frac{\varepsilon_0 h^2}{\pi m e^2} = \frac{\pi \varepsilon_0 \hbar^2}{m e^2}$ is Bohr radius NRL pg 14

$$\Rightarrow \quad \Delta E^{(n)}_{SHIFT} = \frac{e^2}{6\pi\varepsilon_0 a_0^3 n^3} \Delta r^2 \quad \text{for the state } |n00\rangle$$

Hence only s-STATES ($\ell = 0$) experience a shift.

To compute Δr^2 we assume the electron experiences the fluctuating electric field $\mathcal{E}^2(\omega)$

$$F = ma \quad \Rightarrow \quad e\,\mathcal{E}(\omega) \overset{,t}{\cong} m \frac{d^2}{dt^2} \Delta r(\omega)$$

Let $\mathcal{E}(\omega, t) = \mathcal{E}(\omega) e^{i\omega t}$

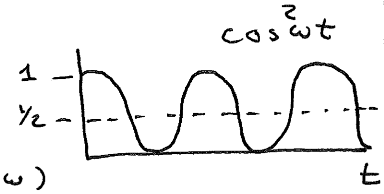
$$\Rightarrow \quad \frac{d^2 \Delta r(\omega,t)}{dt^2} = \frac{e\,\mathcal{E}(\omega)}{m} e^{i\omega t}$$

$$\Rightarrow \quad \Delta r(\omega, t) = -\frac{e\,\mathcal{E}(\omega)}{m\omega^2} e^{i\omega t}$$

Note only the real part of the electric field can contribute to the electron motion: $e^{i\omega t} = \cos\omega t + \cancel{i\sin\omega t}$

$$\Rightarrow \quad \Delta r^2(\omega, t) = \frac{e^2 \mathcal{E}^2(\omega)}{m^2 \omega^4} \cos^2\omega t$$

The time average of $\cos^2\omega t = 1/2$



$$\Rightarrow \quad \langle \Delta r^2(\omega, t)\rangle_{\text{Time}} = \frac{e^2 \mathcal{E}^2(\omega)}{2\,\omega^4} = \Delta r^2(\omega)$$

We can now use expression $\mathcal{E}^2(\omega) = \dfrac{\hbar\omega^3}{\pi^2 c^3 \varepsilon_0} d\omega$

$$\Rightarrow \quad \boxed{\Delta r^2(\omega) = \frac{1}{2} \frac{\hbar e^2}{\pi^2 c^3 \varepsilon_0 m^2} \frac{d\omega}{\omega}}$$

$$\Rightarrow \quad \Delta E_n^{SHIFT}(\omega) = \left(\frac{e^2}{6\pi \varepsilon_0 n^3 a_0^3}\right)\left(\frac{\hbar e^2}{2\pi^2 c^3 \varepsilon_0 m^2}\right) \frac{d\omega}{\omega}$$

Recall pg 14 NRL that Fine structure constant $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c}$

and compton wavelength $\lambda\!\!\!^{-}_c = \dfrac{\hbar}{mc}$

$$\Rightarrow \quad \Delta E_n(\omega) = \frac{\hbar e^4}{12\pi^3 \varepsilon_0^2 c^3 m^2 n^3 a_0^3} \frac{d\omega}{\omega}$$

$$= \frac{\hbar}{12\pi^3 \varepsilon_0^2 c^3 n^3 a_0^3} \left[\frac{\hbar^2}{m^2 c^2} \frac{c^2}{\hbar^2}\right]\left[\frac{e^4}{(4\pi\varepsilon_0\hbar c)^2}(4\pi\varepsilon_0\hbar c)^2\right] \frac{d\omega}{\omega}$$

$$= \frac{\hbar}{12\pi^3 \varepsilon_0^2 c^3 n^3 a_0^3} \frac{c^2}{\hbar^2} \lambda\!\!\!^{-}_c{}^2 \alpha^2 (4\pi\varepsilon_0\hbar c)^2 \frac{d\omega}{\omega}$$

$$= \frac{16\pi^2 \cancel{\varepsilon_0^2} \hbar^3 c^4}{12\pi^3 \cancel{\varepsilon_0^2} \hbar^2 c^3 n^3 a_0^3} \alpha^2 \lambda\!\!\!^{-}_c{}^2 \frac{d\omega}{\omega}$$

$$= \frac{4}{3\pi} \frac{\hbar c}{n^3 a_0^3} \alpha^2 \lambda\!\!\!^{-}_c{}^2 \frac{d\omega}{\omega} \qquad \left(\hbar = \frac{h}{2\pi}\right)$$

$$= \frac{2}{3} \alpha^2 \lambda\!\!\!^{-}_c{}^2 \frac{hc}{n^3 a_0^3} \frac{d\nu}{\nu} \qquad \left(\frac{d\omega}{\omega} = \frac{d\nu}{\nu}\right)$$

$$(2\pi\nu = \omega)$$

$= $ Eq. 2.168 ☺

The total shift is obtained by integrating over all frequecies but $\int_0^\infty \frac{d\omega}{\omega} = \ln(\infty) - \ln(0) = \infty + \infty = \infty$.

So we must cut off the integral.

$$\int_0^\infty \frac{d\omega}{\omega} \rightarrow \int_{\Omega_L}^{\Omega_U} \frac{d\omega}{\omega}$$

At $\Omega_L \to 0$ the rapidly orbiting electron is out of resonance

so $\Omega_L = \omega_0 = 2\pi\nu_0 = \left(\frac{e}{4\pi\varepsilon_0}\right)^2 \frac{2\pi}{\hbar a_0 n^3}$ orbit frequency in nth state

$$= \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right) \frac{\hbar c}{4\pi\varepsilon_0} \frac{2\pi}{\hbar a_0 n^3} = \alpha\left(\frac{c}{2\varepsilon_0 a_0 n^3}\right)$$

For upper limit $\hbar\omega \le mc^2$ to avoid relativity

$$\Rightarrow \boxed{\Omega_U = mc^2/\hbar}$$

Hence $\Delta E_n \equiv \int_{\Omega_L}^{\Omega_U} \Delta E_n(\omega)\, d\omega$

$$= \frac{4}{3\pi} \frac{\hbar c \alpha^2 \lambda^2}{n^3 a_0^3} \int_{\Omega_L}^{\Omega_U} \frac{d\omega}{\omega} = \frac{4}{3\pi} \frac{\hbar c\, \alpha^2 \lambda^2}{n^3 a_0^3} \ln\left[\frac{\Omega_U}{\Omega_L}\right]$$

$$= \frac{4}{3\pi} \frac{\hbar c}{n^3} \frac{\alpha^2 \lambda^2}{n^3 a_0^3} \ln\left[\frac{mc^2}{\hbar\omega_0}\right]$$

For 2S $n=2$ $\Rightarrow \hbar\omega_0 = \frac{\hbar c \alpha}{16\varepsilon_0 a_0^3}$

$$\Delta E_{2S} = \frac{1}{6\pi} \frac{\hbar c\, \alpha^2 \lambda^2}{a_0^3} \ln\left[\frac{mc^2}{\hbar\omega_0}\right]$$

$$= 9.06177 \times 10^{-25}\ \mathrm{J}$$

$$\frac{\Delta E_{2S}}{h} = \boxed{1368\ \mathrm{MHz}}$$

$$\frac{\Delta E_{2S}}{h}(\exp) = 1058\ \mathrm{MHz}$$

error of 30%