A.1 Density operator for pure quantum state

$$\hat{\rho} \equiv |\psi\rangle\langle\psi| \qquad \hat{\rho}^2 = \hat{\rho}$$

Dens. Op. for stat. mixture of quantum states

$$\hat{\rho} \equiv \sum_i \underset{\uparrow}{p_i} |\psi_i\rangle\langle\psi_i|$$

prob system is in state $|\psi_i\rangle$

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}$$

$$0 \le p_i \le 1 \qquad \sum_i p_i = 1 \qquad \sum_i p_i^2 \le 1$$

suppose $\{|n\rangle\}$ orthonormal basis $\langle n|m\rangle = \delta_{nm}$

and $\sum_n |n\rangle\langle n| = \hat{\mathbb{1}}$

$$\Rightarrow \quad |\psi_i\rangle = \sum_n c_n^{(i)} |n\rangle = \sum_n \langle n|\psi_i\rangle |n\rangle$$

matrix elt of $\hat{\rho}$ are $\langle n|\hat{\rho}|m\rangle = \langle n| \sum_i p_i |\psi_i\rangle\langle\psi_i|m\rangle$

$$= \sum_i p_i \langle n|\psi_i\rangle\langle\psi_i|m\rangle = \sum_i p_i \langle n|\psi_i\rangle\langle n|\psi_i\rangle^*$$

$$= \sum_i c_n^{(i)} c_m^{*(i)} p_i$$

$$\mathrm{Tr}[\hat{\rho}] = \sum_n \langle n|\hat{\rho}|n\rangle = \sum_{ni} p_i \langle\psi_i|n\rangle\langle n|\psi_i\rangle$$

$$= \sum_i p_i \langle\psi_i| \underbrace{\{ \sum_n |n\rangle\langle n| \}}_{\mathbb{1}} |\psi_i\rangle = \sum_i p_i = 1$$

$\mathrm{Tr}[\hat{\rho}^2] = 1$ pure

$\mathrm{Tr}[\hat{\rho}^2] < 1$ mixed

mixed ensemble average $\langle\langle \hat{O} \rangle\rangle \equiv \sum_i \underset{\uparrow}{p_i} \langle\psi_i| \underset{\uparrow}{\hat{O}} |\psi_i\rangle$

STAT MECH QUANTUM

$$\Rightarrow \boxed{\langle\langle \hat{O} \rangle\rangle = \mathrm{Tr}(\hat{\rho}\hat{O})}$$

Thermal Fields

single mode in perfect real cavity in equilibrium with walls

temp T, $\frac{\omega}{c} = k$, $\hbar\omega \equiv \varepsilon$

$\frac{1}{k_B T} = \beta$ Boltzman Factor

$\uparrow$ Boltzman const

P_n = prob. nth level is excited S.H.O.

= prob n photons in box

$$= \frac{\exp[-E_n \beta]}{\sum_n \exp[-E_n \beta]}$$

$$\beta \equiv \frac{1}{k_B T}$$

$$E_n = \hbar\omega[n + 1/2] = n\varepsilon + 1/2\,\varepsilon$$

$$\Rightarrow \hat{\rho}_{th} \equiv \frac{\exp[-\beta\hat{H}]}{\mathrm{Tr}[\exp(-\beta\hat{H})]}$$

$$\hat{H} = \hbar\omega(\hat{n} + 1/2) \equiv \varepsilon(\hat{n} + 1/2)$$

$\mathrm{Tr}[\exp[-\beta\hat{H}]] \equiv Z$ partition Fcn

$$= \sum_n \langle n| \exp[-\beta\hat{H}] |n\rangle$$

$$= \sum_{nm} \langle n| \frac{(-\beta\hat{H})^m}{m!} |n\rangle$$

$$= \sum_{nm} \frac{(-\beta\langle n|\hat{H}|n\rangle)^m}{m!}$$

$$= \sum_{nm} \frac{(-\beta E_n)^m}{m!} = \sum_n e^{-\beta E_n} = \overset{\text{vacuum}}{\overset{\downarrow}{e^{-\beta\varepsilon/2}}} \sum_n e^{-n\varepsilon\beta}$$

$$= e^{-\frac{\varepsilon\beta}{2}} \sum_n (\underbrace{e^{-\varepsilon\beta}}_{R})^n = e^{-\varepsilon/2} \sum_{n=0}^{\infty} R^n$$

but $R < 1$ Geometric Series

$$= e^{-\varepsilon/2} \frac{1}{1-R} = \frac{e^{-\varepsilon\beta/2}}{1 - e^{-\varepsilon\beta}} = \boxed{\frac{e^{\varepsilon\beta/2}}{e^{\varepsilon\beta} - 1}} = Z \quad \text{erg/level}$$

$$P_n \equiv \langle n|\hat{\rho}_{Th}|n\rangle = \frac{e^{-\beta E_n}}{Z} = e^{-\beta\varepsilon n}\, e^{-\beta\varepsilon/2}\, \frac{(1-e^{-\varepsilon\beta})}{e^{-\beta\varepsilon/2}}$$

$$= \boxed{(1-e^{-\beta\varepsilon})\, e^{-\beta\varepsilon n}}$$ Prob nth mode atw excited

Note $$\hat{\rho}_{Th} \equiv \hat{\mathbb{1}}\,\hat{\rho}_{Th}\,\hat{\mathbb{1}} = \overbrace{\Big(\sum_n |n\rangle\langle n|\Big)}^{\mathbb{1}}\ \hat{\rho}_{Th}\ \overbrace{\Big(\sum_m |m\rangle\langle m|\Big)}^{\mathbb{1}}$$ ← complete →

$$= \sum_{nm} \langle n|\hat{\rho}|m\rangle\, |n\rangle\langle m| = \frac{1}{Z}\sum_{nm} \exp[-\beta \underbrace{\langle n|\hat{H}|m\rangle}_{E_n\delta_{nm}}]\, |n\rangle\langle m|$$

$$= \frac{1}{Z}\sum_n \exp[-\beta E_n]\, |n\rangle\langle n| \equiv \boxed{\sum_n P_n\, |n\rangle\langle n|}$$

This is the expansion of op $\hat{\rho}_{Th}$ in basis $\{|n\rangle\}$.

Average photon number at freq. ω and temp T:

$$\bar{n} \equiv \langle\hat{n}\rangle = Tr\{\hat{n}\hat{\rho}\} \equiv \sum_n \langle n|\hat{n}\hat{\rho}|n\rangle$$

$$= \sum_{nm} \langle n|\hat{n}\underbrace{|m\rangle\langle m|}_{\mathbb{1}}\hat{\rho}|n\rangle = \sum_{nm} n\,\delta_{nm}\langle m|\hat{\rho}|n\rangle$$

$$= \sum_n n\,\langle n|\hat{\rho}|n\rangle = \underbrace{\sum_n n\,P_n}_{\text{STANDARD DEF } \bar{n} \text{ with prob. dist. } P_n} = (1-e^{-\beta\varepsilon})\sum_n n\,e^{-\beta\varepsilon n}$$

Let's define dimensionless $x \equiv \beta\varepsilon \equiv \hbar\omega/k_BT$

and $r \equiv e^{-x} < 1 \quad \forall\ x>0$

Recall geometric series: $$\sum_{n=0}^{\infty} r^n \equiv \frac{1}{1-r} \quad \forall |r|<1$$

$$\Rightarrow \sum_{n=0}^{\infty} n\,r^n = \sum_{n=1}^{\infty} n\,r^n = \sum_{n-1=0}^{\infty} n\,r^n \quad \begin{matrix}(m=n-1)\\(n=m+1)\end{matrix}$$

$$= \sum_{m=0}^{\infty}(m+1)\,r^{m+1} = r\sum_{m=0}^{\infty}(m+1)\,r^m = r\sum_{m=0}^{\infty}\frac{d}{dr}r^{m+1}$$

$$= r\frac{d}{dr}\sum_{m=0}^{\infty}r^{m+1} = r\frac{d}{dr}r\sum_{m=0}^{\infty}r^m = r\frac{d}{dr}\left[\frac{r}{1-r}\right]$$

$$= r\left[\frac{r'(1-r)-(1-r)'r}{(1-r)^2}\right] = r\left[\frac{(1-\cancel{r})-(0-1)\cancel{r}}{(1-r)^2}\right] = \boxed{\frac{r}{(1-r)^2}}$$

$$\Rightarrow \bar{n} = (1-e^{-x})\sum_n n(\overbrace{e^{-x}}^{r})^n = \frac{(1-r)(r)}{(1-r)^2} = \frac{(r-1)}{(1-r)^2} = \frac{+r}{1-r}$$

$$= \frac{-r}{r-1} = \frac{-1}{1-r^{-1}} = \frac{1}{r^{-1}-1} = \boxed{\frac{1}{e^x-1} = \frac{1}{e^{\frac{\hbar\omega}{kT}}-1}}$$

Hence if $kT >> \hbar\omega \Rightarrow x << 1$ (Rayleigh–Jeans Limit)

$$\Rightarrow \bar{n} = \frac{1}{e^x - 1} \cong \frac{1}{1 + x - 1} = \frac{1}{x} = \boxed{\frac{k_B T}{\hbar\omega}}$$

But if $kT << \hbar\omega \Rightarrow x >> 1$ (Wien Limit)

$$\Rightarrow \bar{n} = \frac{1}{e^x - 1} \approx \frac{1}{e^x} \approx e^{-x} \approx \boxed{e^{-\frac{\hbar\omega}{kT}}}$$

Note: $\bar{n} = \frac{1}{r^{-1} - 1} = \frac{1}{e^x - 1} = \boxed{\frac{r}{1-r}}$

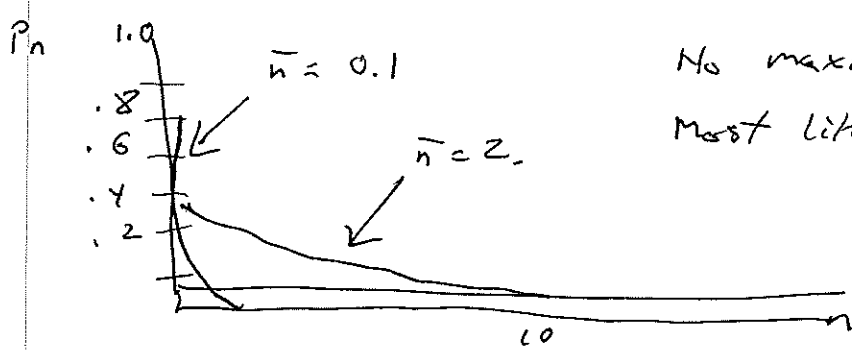
$$\Rightarrow \bar{n}(1-r) = r = \bar{n} - r\bar{n} \Rightarrow r(1+\bar{n}) = \bar{n}$$

$$\Rightarrow r = e^{-x} = \boxed{e^{-\frac{\hbar\omega}{kT}} = \frac{\bar{n}}{\bar{n}+1}}$$

$$\Rightarrow P_n = (1 - r)\, r^n = \left(1 - \frac{\bar{n}}{\bar{n}+1}\right)\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$$

$$= \frac{(\cancel{\bar{n}} + 1 - \cancel{\bar{n}})\,\bar{n}^n}{(\bar{n}+1)^{n+1}} = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}$$

$$\Rightarrow \boxed{\hat{\rho} \equiv \sum_n P_n |n\rangle\langle n| = \frac{1}{(\bar{n}+1)} \sum_n \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|}$$



No maximum at $n = \bar{n}$

Most likely state $= |VAC\rangle$

Fluctuations

$$\Delta n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$$
$$= \langle \hat{n}^2 \rangle - \bar{n}^2$$

$$\langle \hat{n}^2 \rangle = \sum_n n^2 P_n = \sum_n n^2 (1-r) e^{-xn} = (1-r) \sum_n n^2 r^n \quad *$$

$$= (1-r) \sum_{n=1}^{\infty} n^2 r^n = (1-r) \sum_{\underbrace{n-1=0}_{m}}^{\infty} (m+1)^2 r^{m+1}$$

$$= (1-r) \sum_{m=0}^{\infty} (m^2+2m+1) r^{m+1} \overset{\text{Dummy}}{=} (1-r) \sum_{n=0}^{\infty} (n^2+2n+1) r^{n+1}$$

$$= (1-r) r \sum_{n=0}^{\infty} (n^2 r^n + 2n r^n + r^n) = \boxed{(1-r) r \left[\left(\sum_{n=0}^{\infty} n^2 r^n \right) + \frac{2r}{(1-r)^2} + \frac{1}{1-r} \right]}$$

Trick: $* = (1-r) \sum_n n^2 r^n = (1-r) r \left(\sum_{n=0}^{\infty} n^2 r^n \right) + \frac{2r^2}{(1-r)} + r$

$$= (1-r) r \left(\sum_{n=0}^{\infty} n^2 r^n \right) + \frac{2r^2 + r - r^2}{(1-r)}$$

$$= (1-r) r \left(\sum_{n=0}^{\infty} n^2 r^n \right) + \frac{r^2 + r}{1-r}$$

$$\Rightarrow \left[(1-r) - (1-r) r \right] \left(\sum_{n=0}^{\infty} n^2 r^n \right) = \frac{r(1+r)}{(1-r)}$$

$$\Rightarrow \left[1 - r - r - r^2 \right] (\Sigma) = \frac{r(1+r)}{(1-r)}$$

$$\Rightarrow (1-r)^2 (\Sigma) = \frac{r(1+r)}{(1-r)}$$

$$\Rightarrow \boxed{(1-r) \sum_{n=0}^{\infty} n^2 r^n} = \boxed{\frac{r(1+r)}{(1-r)^2} = \langle \hat{n}^2 \rangle}$$

$*$

Hence $\langle \hat{n}^2 \rangle = \left(\frac{r}{1-r} \right) \left(\frac{1}{1-r} + \frac{r}{1-r} \right)$

$$\equiv \bar{n} \left(\frac{1}{1 - \frac{\bar{n}}{\bar{n}+1}} + \bar{n} \right)$$

$$= \bar{n}^2 + \frac{\bar{n}(\bar{n}+1)}{\cancel{\bar{n}}+1-\cancel{\bar{n}}} = \bar{n}^2 + \bar{n}^2 + \bar{n}$$

$$= \boxed{2\bar{n}^2 + \bar{n}} \quad \therefore \quad \text{Eq. } 2.147$$

Hence $\Delta n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = 2\bar{n}^2 + \bar{n} - \bar{n}^2 = \boxed{\bar{n}^2 + \bar{n}}$

$$\Rightarrow \boxed{\Delta n = \sqrt{\bar{n} + \bar{n}^2}} \geq \bar{n} \quad \forall\ \bar{n}$$

Fluctuations bigger than ~~ave~~ mean!

Two limits

$\bar{n} \gg 1 \Rightarrow$

$$\Delta n = \bar{n}\sqrt{1 + 1/\bar{n}} \approx \bar{n}\left(1 + \frac{1}{2}\frac{1}{\bar{n}}\right) = \boxed{\bar{n} + 1/2}$$

So Fluctuations are Larger than mean

$$\bar{n} \ll 1 \Rightarrow \sqrt{\bar{n}(1+\bar{n})} = \sqrt{\bar{n} + \cancelto{0}{\bar{n}^2}} \approx \sqrt{\bar{n}}$$

In this limit fluctuations are nearly that of vaccum state $\Delta n \approx \sqrt{\bar{n}} < \bar{n}$

This is all one mode. For Planck's Law Need continuum Limit $\sum_{\vec{k}s} \rightarrow V \int dk\, \rho(k) = V \int dk \frac{k^2}{\pi^2} dk$

$\uparrow$ D.O.M. F. space

Average Erg. All MODES $= V \int d\omega\, \rho(\omega) = \frac{V}{\pi^2 c^2} \int d\omega\, \omega^2$

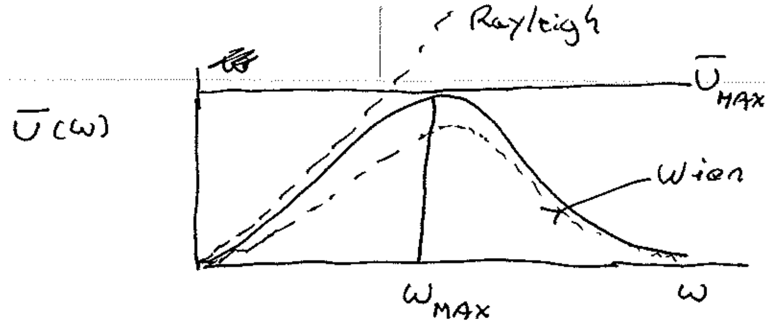
$$\bar{E} = \sum_{ks} \hbar\omega\, \bar{n}(\omega) \rightarrow \frac{V\hbar}{\pi^2 c^2} \int \bar{n}(\omega)\, \omega^3\, d\omega$$

Averag Erg per ω

$$\bar{E}(\omega) = \frac{V\hbar}{\pi^2 c^3} \bar{n}(\omega)\, \omega^3$$

Avg. Erg Density (per volume) per mode = Planck

$$= \bar{U}(\omega) = \frac{\bar{E}(\omega)}{V} = \boxed{\frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}} = \text{B.B.}$$



Set $\bar{U}'(\omega) = 0$ TO FIND ω_{MAX}

$$\bar{U}(\omega) = \frac{\hbar}{\pi^2 c^3}\left(\frac{\hbar\omega}{kT}\right)^3\left(\frac{kT}{\hbar}\right)^3 \frac{1}{e^x - 1}$$

$$= \frac{\beta^{-3}}{c^3(\pi\hbar)^2}\,\frac{x^3}{e^x-1} = \frac{1}{(c\beta)^3(\pi\hbar c)^2} * \frac{x^3}{e^x-1}$$

$$\boxed{x = \frac{\hbar\omega}{kT} \qquad \beta = \frac{1}{kT}}$$

$$\frac{d}{dx}\left[\frac{x^3}{e^x-1}\right] = \frac{3x^2(e^x-1) - x^3e^x}{(e^x-1)^2} = 0$$

$$\Rightarrow \cancel{3x^2(e^x-1) - (e^x-1)^2 = x^3e^x}$$

$$\Rightarrow \cancel{(e^x-1)\left[3x^2 - (e^x-1)\right] = x^3e^x}$$

$$\Rightarrow 3x^2(e^x-1) - x^3e^x = 0 = 3x^2e^x - 3x^2 - x^3e^x = 0$$

$$\Rightarrow (3x^2 - x^3)e^x = 3x^2 \quad\Rightarrow\quad x^2(3-x)e^x = 3x^2$$

$$\Rightarrow \boxed{(3-x)e^x = 3} \quad \text{Transcendental Eq.}$$

$$\boxed{x_{MAX} \cong 2.82}$$

Wien's Displacement Law

$$\Rightarrow \frac{\hbar\omega_{MAX}}{kT} = 2.82 \Rightarrow \boxed{\omega_{MAX} \cong 2.82\,\frac{kT}{\hbar}}$$

$$\Rightarrow \bar{U}_{MAX} = \bar{U}(\omega_{max}) = \frac{(kT)^3}{c^3(\pi\hbar)^2}\,\frac{x_{MAX}^3}{e^{x_{MAX}}-1} = \boxed{\frac{1.42\,(kT)^3}{c^3(\pi\hbar)^2}}$$

Total Erg Density

$$\bar{U}_{TOT} = \int_0^\infty \bar{U}(\omega)\,d\omega = \frac{kT}{\hbar}\int_0^\infty \bar{U}(\omega)\,\frac{\hbar\,d\omega}{kT} = \frac{kT}{\hbar}\left[\int_0^\infty \frac{x^3dx}{e^x-1}\right]\frac{\beta^{-3}}{(\pi\hbar c)^2}$$

$$= \frac{(kT)^4}{c^3\hbar^3(\pi)^2}\underbrace{\int_0^\infty \frac{x^3dx}{e^x-1}}_{\frac{\pi^4}{15}} = \boxed{\frac{1}{15}\frac{\pi^2}{(\hbar c)^3}(kT)^4} \quad \text{Eq. 2.155}$$

Stephan–Boltzman

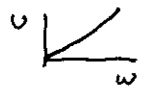
Recovering Rayleigh - Jeans $\hbar\omega \ll kT \Rightarrow x \ll 1$ (waves)

$$\Rightarrow \bar{U}(\omega) = \frac{1}{(\beta c)^3 (\pi\hbar)^2} \frac{x^3}{e^x - 1}$$

$$\approx \frac{1}{(\beta c)^3 (\pi\hbar)^2} \frac{x^3}{\cancel{1} + x - \cancel{1}} \approx \frac{(kT)^3}{c^3 (\pi\hbar)^2} \left(\frac{\hbar\omega}{kT}\right)^2$$

Typo in book

$$\approx \boxed{\frac{kT}{c^3\pi^2}\,\omega^2}$$



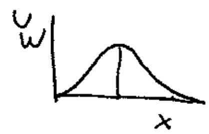
This is Rayleigh - Jeans

Note $\hbar$'s cancelled out $\longrightarrow$ <u>classical limit</u>

$\bar{U}_{RJ}(\omega) \propto \omega^2 \xrightarrow[\omega\to\infty]{} \infty$ ultraviolet catastrophy

Recovering Wien's Law $\hbar\omega \gg kT \Rightarrow x \gg 1$ (Particles)

$$\bar{U}(\omega) = \frac{1}{(\beta c)^3 (\pi\hbar)^2} \frac{x^3}{\underbrace{e^x}_{\gg 1} - \underbrace{1}_{\approx 0}} \approx \left(\frac{kT}{c}\right)^3 \frac{1}{(\pi\hbar)^2} x^3 e^{-x}$$



$$\approx \frac{\cancel{(kT)^3}}{c^3} \frac{1}{(\pi\hbar)^2} \frac{(\hbar\omega)^3}{\cancel{(kT)^3}} e^{-\frac{\hbar\omega}{kT}}$$

$$\approx \boxed{\frac{\hbar\omega^3}{\pi^2 c^3} e^{-\hbar\omega/kT}} = \bar{U}_{\text{wien}}(\omega)$$

agrees with data for large <u>ω</u> particle theory

Note $\frac{d}{dx} x^3 e^{-x} = 3x^2 e^{-x} - x^3 e^{-x}$

$$= (3x^2 - x^3) e^{-x} = 0 = (3 - x)$$

$\Rightarrow x^{\text{wien}}_{\text{MAX}} = 3.0 \approx 2.8$ Hence Wien was a <u>much</u> better theory than Rayleigh - Jeans — no divergences!

$$\int_0^\infty x^3 e^{-x} dx = 6 \approx \frac{\pi^4}{15} = 6.49$$

Even gave finite total energy. Hence <u>most</u> of contribution to Planck's B.B. is due to quantized nature of E&M Field.