Chapter 3

Coherent States

3.1 Problem 3.1

Let assume that the eigenvector of the creation operator $\hat{a}^\dagger$ exists. So we can write

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle. \tag{3.1.1}$$

Now let's write $|\beta\rangle$ as a superposition of the number states, namely

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{3.1.2}$$

Now let's plug the last expression in equation 3.1.1:

$$\begin{aligned}\hat{a}^\dagger|\beta\rangle &= \sum_{n=0}^{\infty} c_n\sqrt{n+1}|n+1\rangle \qquad (3.1.3)\\ &= \beta\sum_{n=0}^{\infty} c_n|n\rangle. \qquad (3.1.4)\end{aligned}$$

From the last express we deduce that

$$\begin{aligned}c_0 &= 0, \qquad (3.1.5)\\ c_{n+1} &= \frac{1}{\beta}c_n\sqrt{n+1}, \qquad (3.1.6)\end{aligned}$$

which means all c_n's $= 0$.

3.2 Problem 3.2

Using equation (3.29), we can determine $\Delta\phi$ for large $|\alpha|$.

$$(\Delta\phi)^2 = \left\langle \phi^2 \right\rangle - (\langle\phi\rangle)^2 \tag{3.2.1}$$

For large α

$$\mathcal{P}(\phi) = \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \exp\left[-2|\alpha|^2(\phi-\theta)^2\right]$$

$$\begin{aligned}
\left\langle \phi^2 \right\rangle &= \int_{-\pi}^{\pi} \phi^2 \mathcal{P}(\phi) d\phi \\
&= \int_{-\infty}^{\infty} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi^2 \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2(2|\alpha|^2)^{3/2}} \\
&= \frac{1}{2|\alpha|^2}
\end{aligned}$$

$$\begin{aligned}
\langle \phi \rangle &= \int_{-\pi}^{\pi} \phi \mathcal{P}(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= \int_{-\infty}^{\infty} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= 0
\end{aligned}$$

$$\Delta\phi = \frac{1}{\sqrt{2|\alpha|^2}},$$

where taking the limit of integration to $\pm\infty$ is justified.

3.3 Problem 3.3

We know that the generating function of the Hermite polynomials is defined as (see for example Arfken):

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{3.3.1}$$

Eq.(3.46) reads

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{\sqrt{2}})^n}{n!} H_n(\xi). \tag{3.3.2}$$

Replacing x, by ξ and t by $\frac{\alpha}{\sqrt{2}}$, we'll get

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} e^{-(\frac{\alpha}{\sqrt{2}})^2+2(\frac{\alpha}{\sqrt{2}})\xi}. \tag{3.3.3}$$

Completing the square in the last exponent by adding and subtracting $\frac{\xi^2}{2}$ we would get the needed result:

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} e^{\frac{\xi^2}{2}} e^{-(\xi-\frac{\alpha}{\sqrt{2}})^2}. \tag{3.3.4}$$

3.4 Problem 3.4

First, we expand $|\alpha\rangle\langle\alpha|$ in number states as

$$|\alpha\rangle\langle\alpha| = \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m|, \tag{3.4.1}$$

so now we can calculate

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = \hat{a}^\dagger \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \tag{3.4.2}$$

$$\begin{aligned}
\hat{a}^\dagger|\alpha\rangle\langle\alpha| &= \hat{a}^\dagger \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \\
&= \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{(n+1)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n+1)|n+1\rangle\langle m| \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-|\alpha|^2} \frac{\alpha^{n-1}}{\sqrt{(n)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n)|n\rangle\langle m|.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left(\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha| &= \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| \\
&= \alpha^*\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| - \alpha^*\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| \\
&+ \sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\sqrt{n+1}|n+1\rangle\langle m| \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-|\alpha|^2}\frac{\alpha^{n-1}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}n|n\rangle\langle m|.
\end{aligned}$$

Notice that we have used $|\alpha|^2 = \alpha\alpha^*$. Also α and α^* are treated linearly independent. The same way, we can prove the other identity.

3.5 Problem 3.5

The quadrature operators are defined in equations (2.52) and (2.53) as

$$\begin{aligned}
\hat{X}_1 &= \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \\
\hat{X}_2 &= \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)
\end{aligned}$$

Using the following properties of the coherent state

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \alpha|\alpha\rangle, \\
\langle\alpha|\hat{a}^\dagger &= \alpha^*\langle\alpha|,
\end{aligned}$$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*) \tag{3.5.1}$$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*) \tag{3.5.2}$$

$$\begin{aligned}
\langle\alpha|\hat{X}_1|\alpha\rangle^2 &= \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right) \\
\langle\alpha|\hat{X}_2|\alpha\rangle^2 &= \frac{-1}{4}\left(\alpha^2 + \alpha^{*2} - 2|\alpha|^2\right)
\end{aligned}$$

$$
\begin{aligned}
\hat{X}_1^2 &= \frac{1}{4}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger) \\
&= \frac{1}{4}(\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}) \\
\hat{X}_1^2 &= \frac{1}{4}(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1) \\
\hat{X}_2^2 &= \frac{-1}{4}(\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^\dagger\hat{a}-1)
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1^2|\alpha\rangle &= \frac{1}{4}(\alpha^2+\alpha^{*2}+2|\alpha|^2+1) \\
\langle\alpha|\hat{X}_2^2|\alpha\rangle &= \frac{-1}{4}(\alpha^2+\alpha^{*2}-2|\alpha|^2-1).
\end{aligned}
$$

Quantum fluctuations of the quadrature operators can be characterized by the variance

$$\left\langle\left(\triangle\hat{X}_{\substack{2\\1}}\right)^2\right\rangle = \left\langle\hat{X}^2_{\substack{2\\1}}\right\rangle - \left\langle\hat{X}_{\substack{2\\1}}\right\rangle^2 . \tag{3.5.3}$$

From the previous equations we will have

$$\left\langle\left(\triangle\hat{X}_1\right)^2\right\rangle_\alpha = \frac{1}{4} = \left\langle\left(\triangle\hat{X}_2\right)^2\right\rangle_\alpha , \tag{3.5.4}$$

which is exactly the same fluctuations for the quadrature operators for the vacuum.

3.6 Problem 3.6

In order to calculate the factorial moments,

$$\langle\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1)\rangle , \tag{3.6.1}$$

for a coherent state $|\alpha\rangle$, one needs to write the operator $\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1)$ in the normal order (all $\hat{a}^\dagger$'s on the left). The claim is that

$$\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1) = \hat{a}^{\dagger r}\hat{a}^r, \tag{3.6.2}$$

which can be proved using the boson commutation rule, $[\hat{a},\hat{a}^\dagger] = 1$, and mathematical induction. Now it is easy to the calculate the factorial moments for a coherent state.

$$\langle\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1)\rangle = |\alpha|^{2r} \tag{3.6.3}$$

3.7 Problem 3.7

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{3.7.1}$$

$$\alpha = |\alpha|e^{i\theta} \tag{3.7.2}$$

$$\hat{C} = \frac{1}{2}\left(\hat{E}+\hat{E}^\dagger\right), \text{ and } \hat{S} = \frac{1}{2i}\left(\hat{E}-\hat{E}^\dagger\right)$$

$$\begin{aligned}
\langle\alpha|\,\hat{E}\,|\alpha\rangle &= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\,\langle n|\,\hat{E}\,|n'\rangle\\
&= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\langle n|m\rangle\langle m+1|n'\rangle\\
&= \alpha e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{C}\,|\alpha\rangle &= \frac{1}{2}\langle\alpha|\left(\hat{E}+\hat{E}^\dagger\right)|\alpha\rangle\\
&= \frac{1}{2}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle + \langle\alpha|\,\hat{E}^\dagger\,|\alpha\rangle\right)\\
&= \frac{1}{2}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle + \langle\alpha|\,\hat{E}\,|\alpha\rangle^*\right)\\
&= \frac{1}{2}\left(\alpha+\alpha^*\right)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}\\
&= \Re(\alpha)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}\\
&= \cos(\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{S}\,|\alpha\rangle &= \frac{1}{2i}\,\langle\alpha|\left(\hat{E}-\hat{E}^{\dagger}\right)|\alpha\rangle \\
&= \frac{1}{2i}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle - \langle\alpha|\,\hat{E}^{\dagger}\,|\alpha\rangle\right) \\
&= \frac{1}{2i}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle - \langle\alpha|\,\hat{E}\,|\alpha\rangle^{*}\right) \\
&= \frac{1}{2i}\left(\alpha-\alpha^{*}\right)e^{-|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \Im(\alpha)e^{-|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \sin(\theta)e^{-|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}}
\end{aligned}$$

$$\begin{aligned}
\hat{C}^{2} &= \frac{1}{4}\left(\hat{E}+\hat{E}^{\dagger}\right)\left(\hat{E}+\hat{E}^{\dagger}\right) \\
&= \frac{1}{4}\left(\hat{E}^{2}+\hat{E}^{\dagger 2}+\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\right)
\end{aligned}$$

$$\begin{aligned}
\hat{S}^{2} &= \frac{-1}{4}\left(\hat{E}-\hat{E}^{\dagger}\right)\left(\hat{E}-\hat{E}^{\dagger}\right) \\
&= \frac{-1}{4}\left(\hat{E}^{2}+\hat{E}^{\dagger 2}-\hat{E}\hat{E}^{\dagger}-\hat{E}^{\dagger}\hat{E}\right)
\end{aligned}$$

$$\begin{aligned}
\hat{E}^{2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|n\rangle\langle n+1|m\rangle\langle m+1| \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n+2|, \\
\hat{E}^{\dagger 2} &= \sum_{n=0}^{\infty}|n+2\rangle\langle n| \\
\hat{E}\hat{E}^{\dagger} &= 1, \\
\hat{E}^{\dagger}\hat{E} &= 1-|0\rangle\langle 0| \\
\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E} &= 2-|0\rangle\langle 0|
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{E}^2\,|\alpha\rangle &= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\,\langle n|\,\hat{E}^2\,|n'\rangle \\
&= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\langle n|m\rangle\langle m+2|n'\rangle \\
&= \alpha^2 e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{C}^2\,|\alpha\rangle &= \frac{1}{4}\,\langle\alpha|\left(\hat{E}^2+\hat{E}^{\dagger 2}+\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\right)|\alpha\rangle \\
&= \frac{1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^{\dagger 2}\,|\alpha\rangle+\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^2\,|\alpha\rangle^*+\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{1}{4}\left(2\Re(\alpha^2)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}+2+e^{-|\alpha|^2}\right) \\
&= \frac{1}{4}\left(2\cos(2\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}}+2+e^{-|\alpha|^2}\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{S}^2\,|\alpha\rangle &= \frac{-1}{4}\,\langle\alpha|\left(\hat{E}^2+\hat{E}^{\dagger 2}-\hat{E}\hat{E}^{\dagger}-\hat{E}^{\dagger}\hat{E}\right)|\alpha\rangle \\
&= \frac{-1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^{\dagger 2}\,|\alpha\rangle-\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{-1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^2\,|\alpha\rangle^*-\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{-1}{4}\left(2\Re(\alpha^2)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}-2-e^{-|\alpha|^2}\right) \\
&= \frac{-1}{4}\left(2\cos(2\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}}-2-e^{-|\alpha|^2}\right)
\end{aligned}$$

As $|\alpha| \to \infty$

$$\begin{aligned}
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2} &= 0\\
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}} &= 1\\
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}} &= 1\\
\langle\alpha|\,\hat{C}\,|\alpha\rangle &= \cos\theta\\
\langle\alpha|\,\hat{S}\,|\alpha\rangle &= \sin\theta
\end{aligned}$$

and

$$\begin{aligned}
\langle\alpha|\,\hat{C}^2\,|\alpha\rangle &= \frac{1}{2}\left(\cos(2\theta)+1\right) = \cos^2(\theta)\\
\langle\alpha|\,\hat{S}^2\,|\alpha\rangle &= \frac{1}{2}\left(\cos(2\theta)-1\right) = \sin^2(\theta)
\end{aligned}$$

$$\begin{aligned}
\left\langle\alpha\left|\left(\Delta\hat{C}\right)^2\right|\alpha\right\rangle &= \left\langle\alpha\left|\hat{C}^2\right|\alpha\right\rangle - \left\langle\alpha\left|\hat{C}\right|\alpha\right\rangle^2 = 0\\
\left\langle\alpha\left|\left(\Delta\hat{S}\right)^2\right|\alpha\right\rangle &= \left\langle\alpha\left|\hat{S}^2\right|\alpha\right\rangle - \left\langle\alpha\left|\hat{S}\right|\alpha\right\rangle^2 = 0
\end{aligned}$$

The uncertainty products of Eqs. (2.215) and (2.216) equalize as $|\alpha| \to \infty$.

3.8 Problem 3.8

a. Let define $|z\rangle$ as

$$|z\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{3.8.1}$$

The eigenvalue equation

$$\begin{aligned}
\hat{E}|z\rangle = z|z\rangle &= \sum_{n=0}^{\infty} zc_n|n\rangle \\
\frac{1}{\sqrt{\hat{n}+1}}\hat{a}|z\rangle &= \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{\hat{n}+1}}\sqrt{n}|n-1\rangle \\
&= \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{n}}\sqrt{n}|n-1\rangle \\
&= \sum_{n=0}^{\infty} c_n|n-1\rangle \\
&= \sum_{n=0}^{\infty} c_{n+1}|n\rangle
\end{aligned}$$

leads to

$$c_n = c_{n-1}z = ... = c_0 z^n. \tag{3.8.2}$$

Thus the eigenstate has the the expansion

$$|z\rangle = \sum_{n=0}^{\infty} c_0 z^n|n\rangle. \tag{3.8.3}$$

The state of Eq. 3.8.3 is normalized for any z, such that $|z| < 1$. For such a case, c_0 can be determined as

$$1 = |c_0|^2 \sum_{n=0}^{\infty} |z|^{2n} = |c_0|^2 \frac{1}{1-|z|^2}, \tag{3.8.4}$$

where we have used the properties of the geometric series. Finally, c_0 and $|z\rangle$ can be defined respectively as

$$\begin{aligned}
c_0 &= \sqrt{1-|z|^2} \\
|z\rangle &= \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n|n\rangle.
\end{aligned}$$

Notice that $|z| < 1$, otherwise the state will not be normalized.

b.

$$\begin{aligned}
\int d^2z|z\rangle\langle z| &= \int d^2z\left(1-|z|^2\right)\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} z^n z^{n'}|n\rangle\langle n'| \\
&= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\int_0^1 d|z|^2\int_0^{2\pi}d\phi\left(1-|z|^2\right)|z|^{2(n+n')}e^{i\phi(n-n')}|n\rangle\langle n'| \\
&= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\int_0^1 dr\,(1-r)\,r^{(n+n')/2}2\pi\delta_{n,n'}|n\rangle\langle n'| \\
&= 2\pi\sum_{n=0}^{\infty}\int_0^1 dr\left(r^n-r^{n+1}\right)|n\rangle\langle n| \\
&= 2\pi\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|n\rangle\langle n|,
\end{aligned}$$

It does not resolve unity.

c. We have proved that the state is not normalized for $|z|<1$. Thus we drop the normalization constant and we write $z=e^{i\phi}$ and we obtain the phase states $|\phi\rangle$ of Eq. (2.221). Obviously the the last states resolve unity as in Eq. (2.223).

d. The average photon number

$$\begin{aligned}
\bar{n} &= \langle z|\hat{n}|z\rangle \\
&= \left(1-|z|^2\right)\sum_{n=0}^{\infty} n|z|^{2n} \\
&= \left(1-|z|^2\right)\frac{\partial}{\partial|z|^2}\sum_{n=0}^{\infty}|z|^{2n} \\
&= \left(1-|z|^2\right)\frac{\partial}{\partial|z|^2}\left(\frac{1}{1-|z|^2}\right) \\
&= \frac{1}{1-|z|^2}
\end{aligned}$$

The photon number distribution for $|z\rangle$ is

$$\begin{aligned}
P_n = |\langle n|z\rangle|^2 &= \left(1-|z|^2\right)|z|^{2n} \\
&= \frac{1}{\bar{n}}\left(\frac{\bar{n}-1}{\bar{n}}\right)^n.
\end{aligned}$$

This distribution resembles the thermal light distribution.

e.

$$\begin{aligned}
\mathcal{P}(\phi) &= \left|\langle\phi|z\rangle\right|^2 \\
&= \left(1-|z|^2\right)\left|\sum_{n=0}^{\infty} e^{in\phi} z^n\right|^2 \\
&= \left(1-|z|^2\right)\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} e^{i(n-n')\phi}|z|^{n+n'}
\end{aligned}$$