Functions of complex variables

1) Functions of real variables
   a) real numbers: a, b, c
   b) real variables: x, y
   c) graphical representation
   d) functions \( f(x) = y \)
   e) basic operations of addition and multiplications
   f) differentiation \( \frac{f(x+dx) - f(x)}{dx} \to \frac{df}{dx} \)
   g) Integration \( S_n = \sum f(x_i)(x_i - x_{i-1}) \), \( S = \lim_{n \to \infty} S_n \)

2) Complex numbers and variables are ordered pairs of real numbers and variables. \((a, b), (x, y)\)

b) Their representation is on the plane

It is convenient to introduce \((1, 0) = 1, (0, 1) = i\) as basis, then \(z = (x, y) = x + iy\) with addition \(z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)\) and multiplication \(z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)\) with \(i \cdot i = -1\)

It is convenient to use \(z = (r \cos \theta, r \sin \theta) = r(\cos \theta + i \sin \theta) = re^{i \theta} = \underbrace{z}_{\text{complex conjugation}}\)

c) Function of complex variables is an pair of ordered real functions \(w(t) = (u(x, y), v(x, y))\) which is a mapping operation

\(z \bigcirc \theta \to w \)
Example of De Moivre's Formula

\[ e^{i\theta} = \cos \theta + i \sin \theta = (\cos \theta + i \sin \theta)^n \] as well

\[ \Rightarrow \cos n\theta = \Re (\cos \theta + i \sin \theta)^n = \Re (\cos^n \theta - \sin^n \theta) \]

\[ \sin 2\theta = 2 \sin \theta \cos \theta \]

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Trigonometric to exponential.

\[ e^{ix} = \cos x + i \sin x \]

\[ e^{-ix} = \cos x - i \sin x \]

\[ \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

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Hypergeometrical functions

\[ \cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2} \]

\[ \cosh(\text{iy}) = \cos y, \quad \sinh(\text{iy}) = i \sin y \]

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\[ \ln z = \ln |z| + \text{arg} z = \ln |z| + \text{arg} z + 2\pi n \]

It is a multi-valued function such as \( \sqrt{x^2} = \pm x \)

\( \sqrt{x^2} = x \) takes only principle value into an account taking \( n=0 \) would give a principle value of \( \ln z \)

\[ \Rightarrow \text{notion of cut lines.} \]
Differentiation.

1. Define \( \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z} \)

The question is what direction \( \Delta z \to 0 \)

\[ f(x) \quad \rightarrow \quad f(x_0) \quad \text{or} \quad f(x) \]

The same is true when \( \tau \) is complex. \( \frac{dw}{d\tau} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} \)

\[ \begin{align*} &\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\text{\Rightarrow Cauchy - Riemann conditions} & \quad \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \\
\end{align*} \]

Function with \( \frac{\partial w}{\partial z} \) in the vicinity of \( z_0 \) is analytic. Function with \( \frac{\partial w}{\partial \bar{z}} \) everywhere is entire.

Example \( f(z) = z^2 \) and \( \bar{f}(z) = \bar{z}^2 \)

Example of application: Preserves topology \( g_{\bar{z}} = \nabla u \)

Consider \( u(x,y) = \text{const} \)

\( \nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \)

\( v = \left( \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right) \) is orthogonal to \( \nabla u \) since \( \langle \nabla u, v \rangle = 0 \).
Integration

\[ \int_{z_0}^{z_0'} f(z) \, dz = \lim_{h \to 0} \sum_{j=1}^{n} f(z_j) (z_j - z_{j-1}) \]

\( f(z) = u + iv \), \( \, dz = dx + idy \).

\[ \int_{z_0}^{z_0'} f(z) \, dz = \int_{x_1, y_1}^{x_2, y_2} u \, dx - v \, dy + i \int_{x_1, y_1}^{x_2, y_2} v \, dx + u \, dy \]

\( \Rightarrow \) complex sum integral = complex sum of real integrals

Example

\[ \oint_{\partial R} \frac{1}{2\pi i} \oint_{C} z^n \, dz = \frac{r^{n+1}}{2\pi i} \int_{0}^{2\pi} \exp (i(n+1)\theta) \, d\theta \]

Cauchy's integral theorem.

\( f \) - analytic \( \Rightarrow \) single valued \( \Rightarrow \) in a simply connected region \( R \) \( \Rightarrow \) for every closed path \( C \) in \( R \)

\[ \oint_{C} f(z) \, dz = \oint_{C} f(z) \, d\sigma = 0 \]

Stoke's theorem

\[ \nabla \times (\mathbf{v} \times \mathbf{x}) + \mathbf{y} \mathbf{v} = \int_{S} \mathbf{v} \cdot (\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial \mathbf{v}}{\partial y}) \, dx \, dy \]

\( \mathbf{S} \)
\[ \int (u \, dx - v \, dy) = - \int \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \, dx \, dy = 0 \]

\[ \int (v \, dx + u \, dy) = \int \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy = 0 \]

Example of motion in potential field does not depend on path. Cauchy-Goursat proof is more general and does not require existence of \( \frac{\partial}{\partial z} \) everywhere in \( \mathbb{R} \) but single simply connected region is still required.

**Multiply Connected Regions**

\[ \Rightarrow \int_{C_1} f(z) \, dz = -\int_{C_2} f(z) \, dz. \]

\[ \Rightarrow \int_{A} f + \int_{EFG} = 0 \quad \text{by Cauchy} \]

\[ \Rightarrow \oint_{C} f = \oint_{C_2} f \]
Cauchy's Integral Formula.

\[ \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0} \, dz = f(z_0) \]

\( f(z) \) - analytic but \( \frac{f(z)}{z - z_0} \) is not at \( z = z_0 \).

\[ \oint_{C_2} \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0) \]

\( f(z) \) function inside region of analyticity is completely defined by its value at the boundary.

Derivatives could be obtained similarly

\[ \frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} = \frac{1}{2\pi i \delta z_0} \left( \oint_{C_2} \frac{f(z)}{z - z_0 - \delta z_0} \, dz - \oint_{C_2} \frac{f(z)}{z - z_0} \, dz \right) \]

\[ \Rightarrow f'(z_0) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z - z_0)^2} \, dz. \]

\[ \Rightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_2} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \]

\( f(z) \) analytic guarantees all derivatives

Morera's theorem without proof

If a function \( f(z) \) is continuous in a simply connected region \( R \) and \( \oint_C f(z) \, dz = 0 \) for every closed contour \( C \) within \( R \), then \( f(z) \) is analytic throughout \( R \).
Cauchy's inequality

If \( f(z) = \sum a_n z^n \) is analytic and bounded, and \( |f(z)| \leq M \) on a circle of radius \( r \) about the origin, then,

\[
|a_n| r^n \leq M
\]

Proof: \( |a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \, dz \right| \leq M(r) \frac{2\pi r}{2\pi r^{n+1}} \)

\( \Rightarrow \) Liouville's theorem.

If \( f(z) \) is analytic and bounded on the complex plane \( \Rightarrow \) it is constant. Since \( a_n \leq M r^{-n} \to 0, n \not= 0 \)

\( \Rightarrow \) the must be singularities so let us learnt how to leave with them. \( \Rightarrow \) Taylor Expansion, Analytical continuation, \( \Rightarrow \) Laurent Expansion.
Taylor Expansion

1) Suppose that we are trying to expand \( f(z) \) near \( z_0 \) and \( z_1 \) is the nearest point where \( f(z) \) is not analytic.

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z_0} \, dz'
\]

\[
\neq \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0) \left[ 1 - \frac{z - z_0}{z' - z_0} \right]} \, dz'
\]

\[
\left| \frac{z - z_0}{z' - z_0} \right| < 1
\]

\[
f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}} \frac{f(z')}{z' - z_0} \, dz'
\]

\[
= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}} \oint_C \frac{f(z')}{z' - z_0} \, dz'
\]

\[
= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n
\]

2) Taylor Expansion defines analytic function \( f \) inside the circle of convergence.

If we choose a point inside \( S_1 \) and expand again in a circle where we extend beyond initial region of \( S_1 \), we have another analytic continuation. Based on this if \( f(z) = g(z) \) on a line or in a region and both \( f(z) \) and \( g(z) \) are analytic then \( f(z) = g(z) \) everywhere.
Example \( f(2) = \frac{1}{1 + 2} \)

It has pole at \( z = -1 \).

Maclaurin series \( \frac{1}{1 + 2} = \sum_{n=0}^{\infty} (-2)^n \) converges \( |z| < 1 \).

Suppose that we Taylor expand around \( z_0 = i \) so that

\[
f(2) = \frac{1}{(1+i) + (2+i)} = \frac{1}{(1+i)(1+\frac{2-i}{2+i})}
= \frac{1}{1+i} \sum (-\frac{2-i}{2+i})^n \text{ converges } |z| < \sqrt{2}\]

could be analytically expanded indefinitely.

However \( \frac{1}{1 + z} \) is a single term Laurent series valid in all plane that we consider next.

Laurent Series

\[
f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{z' - z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{z' - z}
\]

Note that both are traversed in positive sense.

We play same trick by adding and subtracting \( z_0 \)

\[
e \rightarrow \frac{2}{z' - z_0} + z_0 - z \text{ with } |z' - z_0| < |z - z_0| \text{ on } C_2 \text{ and } |z' - z_0| > |z - z_0| \text{ on } C_1
\]

\[
f(2) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^{n+1}} \oint_{C_1} \frac{f(z')dz'}{c_1 (z' - z_0)^n} + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{(z - z_0)^{-n}}{c_2} \oint_{C_2} f(z')dz'
\]
First sum \( S_1 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{dz'}{(z' - z_0)^{n+1}} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \frac{dz'}{n!} \)

converges within \( C_1 \)

Second sum \( S_2 = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z') \, dz' \)

converges outside of \( C_2 \)

\[ \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ with } a_n = \frac{1}{2\pi i} \oint_{C} \frac{f(z') \, dz'}{(z' - z_0)^{n+1}} \]

\[ \Rightarrow \text{ Laurent series.} \]

Example \( f(z) = \frac{1}{z(z-1)} \) and choose \( z_0 = 0 \), \( \Rightarrow \)

\[ a_n = \frac{1}{2\pi i} \oint_{C} \frac{dz'}{(z')^{n+2}(z'-1)} = -\frac{1}{2\pi i} \oint_{C} \frac{dz'}{(z')^{n+2}} \]

\[ a_n = -\frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint_{C} \frac{dz'}{(z')^{n+m+2}} = \begin{cases} -1 & \text{for } n \geq -1 \\ 0 & \text{for } n < -1 \end{cases} \]

\[ \Rightarrow \frac{1}{z(z-1)} = -\frac{1}{z} - \frac{1}{z} - z^2 - z^3 = \sum_{n=-1}^{\infty} z^n \]

which could be confirmed by

Expansion of \( \frac{1}{z-1} \)
Mapping by simple functions

1) Translation \( w = z + z_0 \).

2) Rotation \( w = z \ e^{i\theta} = r_0 \ e^{i\theta_0} \). Let \( r = r_0, \ \theta = \theta + \theta_0 \).

3) Inversion \( w = \frac{1}{z} \) transforms straight lines and circles into straight lines or circles.

Mapping by more complicated functions

Branch Points and Multivalent Functions

1) \( w = z^2 \)

2) The inverse is \( z = \sqrt{w} \) it has the cut line that joins the two branch point singularities at 0 and \( \infty \).