In classical mechanics the symmetry of a physical system leads to conservation laws:
1) translation symmetry $\Rightarrow$ momentum conservation
2) rotation symmetry $\Rightarrow$ angular momentum conservation

Group theory is the mathematical tool to treat invariants and symmetries.

Are you a group?

Mathematically, group $G$ is defined as a set of objects or operations, called as the elements of $G$, that may be combined or "multiplied" to form a well-defined product in $G$ that satisfies the following four conditions:

1. if $a$ and $b \in G \Rightarrow$ the product $ab$ is also element of $G$
2. This multiplication is associative
3. There is a unit element $I \in G$: $Ia = aI = a$
4. There must be an inverse: $a^{-1}$ for any $a$
Example of a group: 1, a, b, c
with a correspondence:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td>b</td>
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<td>a</td>
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<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

counterclockwise coordinate rotations

\[ R(\psi) = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix} \]

with multiplication \[ R(\psi_1, \psi_2) = R(\psi_1) R(\psi_2) \]

\[ \begin{pmatrix} \cos \psi_1 & \sin \psi_1 \\ -\sin \psi_1 & \cos \psi_1 \end{pmatrix} \begin{pmatrix} \cos \psi_2 & \sin \psi_2 \\ -\sin \psi_2 & \cos \psi_2 \end{pmatrix} = \begin{pmatrix} \cos(\psi_1 + \psi_2) & \sin(\psi_1 + \psi_2) \\ -\sin(\psi_1 + \psi_2) & \cos(\psi_1 + \psi_2) \end{pmatrix} \]

It is commutative or Abelian because the order in which these rotations are performed does not matter.

The group name is \( SO(2) \). It has infinitely many elements. It has finite subgroups such as \( \{ R(0), R(\pi) \} \), \( \{ R(0), R(\frac{\pi}{2}), R(\pi), R(\frac{3\pi}{2}) \} \)

\( O(n) \): orthogonal nxn matrices
\( SO(n) \): if \( \det = +1 \), with \( S = "special" \)
\( U(n) \): unitary nxn matrices
\( SU(n) \): if \( \det = +1 \)

These are the classical Lie groups
Homomorphism, Isomorphism.

There may be a correspondence between the elements of two groups $G$ and $H$. $G$ and $H$ are isomorphic.

In the more general case, we would have two-to-one or many-to-one correspondence, thus $G$ and $H$ are homomorphic.

Representations

$\{1, a, b, c\}$ could be represented by $\{1, i, -1, -i\}$ or $\{R(0), R(\pi/2), R(\pi), R(3\pi/2)\}$.

Why matrices are so popular as representations of the group $G = \{R\}$ such that $RHR^{-1} = H$?

$H\psi = E\psi$ and it is invariant under $G = \{R\}$ such that $RHR^{-1} = H$.

Thus $RH\psi = E\psi = RHR^{-1}RV = H(RV)$, which means $\psi$ is an eigenvector for all $RV = \gamma$ degenerate system.
In vector space \( \mathbf{V}_\psi \) of transformed solutions

\[ R\psi_i = \sum_k R_k \psi_k \]

Thus we can associate \( R \) in \( G \) and a matrix \( \psi_k \).

Irreducible representation: if any \( \psi_k \in \mathbf{V}_\psi \), if not all is reached then \( \mathbf{V}_\psi = \mathbf{V}_1 \oplus \mathbf{V}_2 \)

Then it is reducible and one can find \( U \) so that \( U (\psi_k) U^* = \begin{pmatrix} \psi_1 & 0 & \cdots \\ 0 & \psi_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \)

The irreducible representations play a role in the group theory that is roughly analogous to the unit vectors of vector analysis.
Generators of Continuous Groups

A characteristic of continuous groups known as Lie groups is that the parameters of a product element are analytic functions of the parameters of the factors.

From studying of the whole group to studying group elements in the neighborhood of the identity element.

Consider example of $SO(2)$

$$R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = 1_2 \cos \phi + i 0_2 \sin \phi = e^{i\phi}$$

Let us look for an exponential representation

$$R = \exp \left( i \varepsilon S \right), \quad \varepsilon \to 0$$

If $R$ does not change the volume element, that is, $\det R = 1$

$$\Rightarrow \det(R) = \exp \left( \text{tr}(\ln R) \right) = \exp \left( i \varepsilon \text{tr}(S) \right) = 1$$

$\Rightarrow$ generators are traceless, $\text{tr}(S) = 0$.

which is true for $SO(n)$ and $SU(n)$

It could be also shown that $S = S^\dagger \Rightarrow$ Hermitian
\[ R_i^{-1}R_j^{-1}R_iR_j = R_{ij} \]

or
\[ 1 + \epsilon_i \epsilon_j \left[ S_i, S_j \right] = 1 + \epsilon_i \epsilon_j \sum c_{ij}^k S_k + ... \]

Closure relation of the generators of the Lie group \( G \):

\[ \left[ S_i, S_j \right] = \sum c_{ij}^k S_k \quad \text{where} \]

\( c_{ij}^k \) are the structure constants of the group \( G \).

Obviously \( c_{ij}^k = -c_{ij}^k \).

If the commutator is taken as a multiplication law of generators \( \Rightarrow \) vector space of generators becomes an algebra, the Lie algebra \( G \) of the group \( G \).

For \( SU(e+1) \Rightarrow A_e \)

For \( SO(2e+1) \Rightarrow B_e \)

For \( SO(2e) \Rightarrow D_e \)

where \( e \), positive integer, called the rank of the group or of its algebra \( G \).
It holds for all double commutators

$$\left\{ [s_i, s_j], s_k \right\} + \left\{ [s_j, s_k], s_i \right\} + \left\{ [s_k, s_i], s_j \right\} = 0$$

$$\Rightarrow \left\{ C_{ij}^m C_{nk}^m + C_{jk}^m C_{mi}^n + C_{ki}^m C_{mj}^n \right\} = 0$$

$$\Rightarrow$$ basis of Lie algebra from which finite elements of Lie group near its unity can be reconstructed.

Back to symmetry of Hamiltonians of quantum system

$$H = H_R = e^{i\epsilon S} H e^{-i\epsilon S} = H + i\epsilon \left[S, H\right] + \frac{1}{2} \epsilon^2 [S, [S, H]]$$

Barber-Hausdorff formula

$$\Rightarrow [S, H] = 0$$ so for given symmetry $R$

it is its generator that commutes with a Hamiltonian.

$$\Rightarrow$$ simultaneously diagonalized for hermitian common eigenfunctions
degenerate values of $H$ could be labeled by $s$ of $S$.

$$\Rightarrow$$ Group theory (symmetries) lead to separation of variables.
Example of $SO(2)$

\[-i \frac{d R_2(\psi)}{d \psi} \bigg|_{\psi=0} = -i \begin{pmatrix} -\sin \psi & \cos \psi \\ -\cos \psi & -\sin \psi \end{pmatrix} \bigg|_{\psi=0} = \sigma_z\]

Example of $SO(3)$

\[-i \frac{d R_2(\psi)}{2 \psi} \bigg|_{\psi=0} = \sigma_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies R_2 = I_3 + i \delta \psi \sigma_z \]

and similarly

\[S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\]

Relation to the orbital angular momentum

Rotation $R$ rotates the coordinates $\vec{x}' = R \vec{x}$

we define $R$ on $\psi \delta y$

\[R \delta \psi(x, y, z) = \psi'(x, y, z) = \psi(\vec{x}')\]

\[R z(\delta \psi) \psi(x, y, z) = \psi(x, y, z) - \psi(x, y, z) = \frac{\partial^2 \psi}{\partial x^2} \psi + \left(1 + (i \delta \psi \sigma_z) \right) \psi(x, y, z)\]

\[= \frac{d R_2}{\delta \psi} = -i \sigma_z R_2(\psi) \implies R_2(\psi) = \exp(-i \psi \sigma_z)\]
if we recognize that the matrix element

\[ L_z = (x, y, z) S_z \left( \frac{\partial^2}{\partial x^2} \right) \]

it becomes clear why \( L_x, L_y, \) and \( L_z \) satisfy the same commutation relation

\[ [L_i, L_j] = i \epsilon_{ijk} L_k \leq 3 \]

\( S_x, S_y, \) and \( S_z \) and yield the same structure constants

\[ i \epsilon_{ijk} = C_{ij}^k \] of \( SO(3) \)

\[ SU(2) = SO(3) \] Homomorphisms

\( SU(2) \) is of order 2 and depends on three real continuous parameters \( \xi, \eta, \zeta \) which are often called the Cayley-Klein parameters

\[ U_2 (\xi, \eta, \zeta) = \begin{pmatrix} e^{i \xi \cos \eta} & e^{i \xi \sin \eta} \\ -e^{-i \xi \sin \eta} & e^{-i \xi \cos \eta} \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \]

\[ \text{det} = 1, \quad u^+ u = 1 = u u^+ \]

\[ = \gamma U_1 = \exp \left( \frac{i \alpha_1 \sigma_1}{2} \right), \quad U_2 = \exp \left( \frac{i \alpha_2 \sigma_2}{2} \right), \quad U_3 = \exp \left( \frac{i \alpha_3 \sigma_3}{2} \right) \]

based on \( -i \frac{\partial U_2}{\partial \xi} = \sigma_3, \quad -i \frac{\partial U_2}{\partial \eta} = \sigma_2, \quad -i (\sin \eta)^{-1} \frac{\partial U_2}{\partial \zeta} = \sigma_1 \)
\[ s_i = \frac{1}{2} s_i \Rightarrow \{ s_i, s_j \} = \epsilon_{ijk} s_k \]

U_2 performs rotation in 2d complex space:

\[ z' = u_2 z = u_2 (z_2) \]

\[ (z')^* z' = (z)^* z \] thus rotation.

Similarly \( SL(2) \) rotation in 3d real space:

\[ \mathbf{r} = (x, y, z) \Rightarrow x^2 + y^2 + z^2 \text{ is invariant} \]

\[ = x'^2 + y'^2 + z'^2 \]

So we have similar interpretation and same number of independent parameters \( \Rightarrow \) suggests some sort of correspondence. Isomorphic to homomorphism.

\[ \begin{align*}
M' &= U M U^+ , & M &= x \sigma_1 + y \sigma_2 + z \sigma_3 \\
\Rightarrow x^2 + y^2 + z^2 \text{ is invariant as in } S O(3) \\
\end{align*} \]

Let us consider a special case:

\[ U_2 = \begin{pmatrix} e^{i \phi} & 0 \\ 0 & e^{-i \phi} \end{pmatrix} \]

\[ \begin{align*}
\sigma_1 &\rightarrow \cos 2 \phi \sigma_1 - \sin 2 \phi \sigma_2 \\
\sigma_2 &\rightarrow \sin 2 \phi \sigma_1 + \cos 2 \phi \sigma_2 \\
\sigma_3 &\rightarrow \sigma_3 \\
\end{align*} \] rotation with half angle \( \phi = d_2 \)
Orbital angular momentum

In classical mechanics \( L = r \times p \).

In quantum mechanics \( L = -i \hbar \times \partial \)

with \( [\hat{b}_i, \hat{b}_j] = i \delta_{ij} \hbar \).

In addition \( [L^2, \hat{b}_i] = 0 \) since \( L^2 = L_x^2 + L_y^2 + L_z^2 \) is a rotational scalar.

More generally one is interested in angular momentum \( \mathbf{J} \). Example \( J = J_z + \frac{1}{2} \).

From a group theory prospective, one has a system with rotational symmetry and thus generators described by \( \mathfrak{so}(3) \), \( \mathfrak{su}(2) \) or \( \mathfrak{su}(2) \approx \mathfrak{so}(3) \) for \( \mathbf{J} \).

These groups have generators \( \mathbf{J}_i \) such that

\[
[\mathbf{J}_i, \mathbf{J}_j] = i \epsilon_{ijk} \mathbf{J}_k
\]

it could be shown that cassinir invariant \( \mathbf{J} \cdot \mathbf{J} = J^2 = \mathbf{J}_i^2 \)

commutes \( [\mathbf{J}^2, \mathbf{J}_i] = 0 \)

any \( \mathbf{J}_i \) and \( \mathbf{J}^2 \) pair

in this case state of the system is described by both \( \lambda \) and \( M = \lambda, \lambda, M \) such that

\[
\mathbf{J}_z \lambda, M \rangle = \lambda \lambda, M \rangle \quad \text{and}
\]

\[
\mathbf{J}_\pm \lambda, M \rangle = \lambda \lambda, M \pm \rangle
\]

let me introduce

\( \mathbf{J}_+ = \mathbf{J}_x + i \mathbf{J}_y \)

\( \mathbf{J}_- = \mathbf{J}_x - i \mathbf{J}_y \)
then \[ J^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J_z^2 \]

\[ [ J_z, J_+ ] = \pm J_+ , \quad [ J_z, J_- ] = 2 J_z \]

let us look at \( (J+1\lambda, M) \) state.
this state has same \( \lambda \)
this state has \( M = M+1 \) due to \( \mathbf{\mathcal{O}} \)
thus raising raising and lowering operators.

\[ \langle J^2 - J_z^2 \rangle > 0 \Rightarrow \lambda - M^2 > 0 \]

let \( J = \max (M) \) then \( J+1\lambda, J > = 0 \) and \( J-J+1\lambda, J > = 0 \)

but \( S^2 = J_z J_+ + J_+ (J_+ + 1) \)

\[ \Rightarrow \lambda = S(J+1) > 0 \]

similarly \( J' = \min (M) = -S \)

could be shown that

\[ J+1\lambda, M > = \sqrt{(S-M)(S+M+1)} \mid S, M+1 > \]

\[ J-1\lambda, M > = \sqrt{(S+M)(S-M+1)} \mid S, M-1 > \]
Summary of Lie Groups and Lie Algebra.

1) we choose linearly independent and mutually commuting generators \( H_i \), example \( su(2) \) for \( SO(3) \), such that \( [H_i, H_k] = 0 \) ==> then \( l \) is the rank.

2) All other generators \( E_\alpha \) can be shown to be raising and lowering operators

\[ [H_i, E_\alpha] = \pm \alpha_i E_\alpha, \quad \alpha_i = 1, 2, \ldots, l \]

The set of so-called root vectors \( \{E_1, E_2, \ldots, E_l\} \) form the root diagram of the Lie algebra.

3) Commuting \( H_i \) gives us a set of eigenvalues

The set of so-called weight vectors \( \{\lambda_1, \lambda_2, \ldots, \lambda_l\} \) for an irreducible representation form a weight diagram.
operators $L_+$ and $L_-$, explicit forms are given in Exercises 2.5.14 and 12.6.7. The reader can now show (see also Exercise 12.7.2) that

$$
(J M | J_- (J_+ | J M)) = (J_+ | J M)^\dagger J_+ | J M).
$$

(4.82)

As $J_+$ raises the eigenvalue $M$ to $M + 1$, we relabel the resultant eigenfunction $|J M + 1\rangle$. The normalization is given by Eq. (4.80) as

$$
J_+ | J M \rangle = \sqrt{(J - M)(J + M + 1)} |J M + 1\rangle,
$$

(4.83)

taking the positive square root and not introducing any phase factor. By the same arguments

$$
J_- | J M \rangle = \sqrt{(J + M)(J - M + 1)} |J M - 1\rangle.
$$

(4.84)

Finally, as $M$ ranges from $-J$ to $+J$ in unit steps, $2J$ must be an integer; $J$ is either an integer or half of an odd integer. As seen later, orbital angular momentum is described with integral $J$. From the spins of some of the fundamental particles and of some nuclei, we get $J = 1/2, 3/2, 5/2, \ldots$. Our angular momentum is quantized—essentially as a result of the commutation relations.

In spherical polar coordinates $\theta, \varphi$ the functions $\langle \theta, \varphi | l m \rangle = Y^m_l (\theta, \varphi)$ are the spherical harmonics of Section 12.6.

**Summary of Lie Groups and Lie Algebras**

The general commutation relations, Eq. (4.14) in Section 4.2, for a classical Lie group [SO($n$) and SU($n$) in particular] can be simplified to look more like Eq. (4.71) for SO(3) and SU(2) in Section 4.3. Here, we merely review and, as a rule, do not provide proofs for various theorems that we explain.

First, we choose linearly independent and mutually commuting generators $H_i$, which are generalizations of $J_z$ for SO(3) and SU(2). Let $l$ be the maximum number of such $H_i$ with

$$
[H_i, H_k] = 0.
$$

(4.85)

Then $l$ is called the rank of the Lie group $G$ or its Lie algebra $\mathcal{G}$. The rank and dimension or order of some Lie groups are given in Table 4.2. All other generators $E_a$ can be shown to be raising and lowering operators with respect to all the $H_i$ so that

$$
[H_i, E_a] = \alpha_i E_a, \quad i = 1, 2, \ldots, l.
$$

(4.86)

The set of so-called root vectors $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ form the root diagram of $\mathcal{G}$.

Because the $H_i$ commute, they can be simultaneously diagonalized. They provide us with a set of eigenvalues $m_1, m_2, \ldots, m_l$ [projection or additive quantum numbers generalizing $M$ of $J_z$ in SO(3) and SU(2)]. The set of so-called weight vectors $(m_1, m_2, \ldots, m_l)$ for an irreducible representation form a weight diagram. (See Fig. 4.3.)
Chapter 4 Group Theory

Table 4.2 Rank and Order of Unitary and Rotational Groups

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>( A_l )</th>
<th>( B_l )</th>
<th>( D_l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lie group</td>
<td>SU((l + 1))</td>
<td>SO((2l + 1))</td>
<td>SO((2l))</td>
</tr>
<tr>
<td>Rank</td>
<td>( l )</td>
<td>( l )</td>
<td>( l )</td>
</tr>
<tr>
<td>Order</td>
<td>( l(l + 2) )</td>
<td>( l(2l + 1) )</td>
<td>( l(2l - 1) )</td>
</tr>
</tbody>
</table>

There are \( l \) invariant operators \( C_i \), called Casimir operators, which commute with all generators and are generalizations of \( J^2 \):

\[
[C_i, H_j] = 0, \quad [C_i, E_\alpha] = 0, \quad i = 1, 2, \ldots, l. \tag{4.87}
\]

The first one, \( C_1 \), is a quadratic function of the generators, the others are more complicated. Because the \( C_j \) commute with all \( H_j \), they can be simultaneously diagonalized with \( H_j \). Their eigenvalues \( c_1, c_2, \ldots, c_l \) characterize irreducible representations and stay constant while the weight vector varies over any particular irreducible representation. Thus the general eigenfunction may be written as

\[
|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle, \tag{4.88}
\]

generalizing \( |JM\rangle \) of SO(3) and SU(2). Their eigenvalue equations are

\[
H_i|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle = m_i|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle \tag{4.89a}
\]

\[
C_i|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle = c_i|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle. \tag{4.89b}
\]

We can now show that \( E_\alpha |\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle \) has the weight vector \((m_1 + \alpha_1, m_2 + \alpha_2, \ldots, m_l + \alpha_l)\) using the commutation relations, Eq. (4.86), in conjunction with Eqs. (4.89a) and (4.89b):

\[
H_i E_\alpha |\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle \\
= (E_\alpha H_i + [H_i, E_\alpha])|\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle \\
= (m_i + \alpha_i) E_\alpha |\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle. \tag{4.90}
\]

Therefore,

\[
E_\alpha |\(c_1, c_2, \ldots, c_l\)m_1, m_2, \ldots, m_l\rangle \sim |\(c_1, \ldots, c_l\)m_1 + \alpha_1, \ldots, m_l + \alpha_l\rangle,
\]

the generalization of Eqs. (4.83) and (4.84) from SO(3). These changes of eigenvalues by the operator \( E_\alpha \) are called its selection rules in quantum mechanics. They are displayed in the root diagram of a Lie algebra.

Examples of root diagrams are given in Fig. 4.6 for SU(2) and SU(3). If we attach the roots denoted by arrows in Fig. 4.6b to a weight in Figs. 4.3 or 4.5a, we can reach any other state (represented by a dot in the weight diagram).
4.4 Angular Momentum Coupling

\[ \beta \]
\[ \alpha + \beta \]
\[ 60^\circ \]
\[ \alpha \]
\[ -\alpha - \beta \]
\[ -\beta \]

**Figure 4.6** Root diagram for (a) SU(2) and (b) SU(3).

**Exercises**

1. Show that (a) \([J_+, J^2] = 0\), (b) \([J_-, J^2] = 0\).

2. Derive the root diagram of SU(3) in Fig. 4.6b from the generators \(\lambda_i\) in Eq. (4.61).
   
   **Hint.** Work out first the SU(2) case in Fig. 4.6a from the Pauli matrices.

.4 Angular Momentum Coupling

In many-body systems of classical mechanics the total angular momentum is the sum \(L = \sum L_i\) of the individual orbital angular momenta. In quantum mechanics conserved angular momentum arises when particles move in a central potential, such as the Coulomb potential in atomic physics, a shell model potential in nuclear physics, or a confinement potential of a quark model in particle physics. In the relativistic Dirac equation, orbital angular momentum is no longer conserved, but \(J = L + S\) is conserved, the total angular momentum of a particle consisting of its orbital and intrinsic angular momentum, called spin \(S = \sigma/2\) in units of \(\hbar\).

It is readily shown that the sum of angular momentum operators obeys the same commutation relations in Eq. (4.37) or (4.41) as the individual angular momentum operators provided those from different particles commute.

**Clebsch-Gordan Coefficients: SU(2)-SO(3)**

Clearly, combining two commuting angular momenta \(J_i\) to form their sum

\[ J = J_1 + J_2, \quad [J_{1i}, J_{2k}] = 0, \quad (4.91) \]
Figure 4.3 Baryon octet weight diagram for SU(3).

The choice of SU(3) was based first on the two conserved and independent quantum numbers \( H_1 = I_3 \) and \( H_2 = Y \) (i.e., generators with \([I_3, Y] = 0\), not Casimir invariants, see the summary in Section 4.3 on page 264) that call for a group of rank 2. Second, the group had to have an eight-dimensional representation to account for the nearly degenerate baryons and four similar octets for the mesons. In a sense SU(3) is the simplest generalization of SU(2) isospin. Three of its generators are zero-trace Hermitian 3 \( \times \) 3 matrices that contain the 2 \( \times \) 2 isospin Pauli matrices \( \tau_i \) in the upper left corner:

\[
\lambda_i = \begin{pmatrix}
\tau_i & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad i = 1, 2, 3.
\] (4.61)

Thus, the SU(2)-isospin group is a subgroup of SU(3)-flavor with \( I_3 = \lambda_3/2 \). For other generators have the off-diagonal 1 of \( \tau_1 \), and \(-i, i\) of \( \tau_2 \) in all other possible locations to form zero-trace Hermitian 3 \( \times \) 3 matrices

\[
\lambda_4 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad \lambda_5 = \begin{pmatrix}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix},
\]

\[
\lambda_6 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad \lambda_7 = \begin{pmatrix}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}.
\] (4.61)