Everything we know in sub-atomic physics is obtained from scattering exps.

11.1.1 Classical Scattering

\[ b = \text{"impact parameter [meters"]} \]

\[ \theta = \text{scattering angle} \]

\[ \text{Hard sphere radius } R \quad \lambda = \text{incidence angle} \]

\[ \sin[\theta - \phi] = \sin \alpha \cos B - \sin B \cos \phi \]

\[ b = R \sin \left[ \frac{\pi}{2} - \frac{\theta}{2} \right] \]

\[ = R \left[ \sin \frac{\pi}{2} \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \cos \frac{\pi}{2} \right] = R \cos \frac{\theta}{2} = \delta \]
NOTE NO SCATTERING $b > R$ A MISS.

$\Rightarrow \quad \Theta = \begin{cases} 
2 \cos \left[ \frac{b}{R} \right] & ; \quad b \leq R \\
0 & ; \quad b > R
\end{cases}

\begin{array}{c}
b = 0 \Rightarrow \quad \Theta = 2 \cos \left[ \theta \right] = 2 \cdot \pi \rightarrow 180^\circ \text{ REVERSAL DIRECT HIT!}.
\end{array}

\begin{array}{c}
b = R \Rightarrow \quad \Theta = 2 \cos \left[ 1 \right] = 2.0 = 0 \sim 0^\circ \text{ NEAR MISS!}
\end{array}

What $b$ gives $\Theta = 90^\circ$?

$90^\circ \Rightarrow \quad \frac{\pi}{2} = 2 \cos \left[ \frac{1}{R} \right]$

$\Rightarrow \quad \frac{\pi}{4} = \cos \left[ \frac{b}{R} \right]$

$\Rightarrow \quad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{b}{R}$

$\Rightarrow \quad b = \frac{\sqrt{2}}{2} \cdot R \quad R < R$

$\begin{array}{c}
b = \frac{\sqrt{2}}{2} R \quad [ \text{Diagram} ]
\end{array}$
We are typically interested in a beam of incident particles with a spread of impact parameters and corresponding spread of scattering $\theta$.~

Looking down the beam:

Treat differential patch of area ring:

\[
\frac{d\varphi}{b} = \int_{\theta}^{\theta + d\varphi} b \, db \, d\varphi
\]

Area of ring = $[m^2]$ area

\[
\int = \text{flux of incoming particles} = \frac{\# \text{particles}}{[s] \cdot [m^2]}
\]

\[
\int \cdot d\varphi = \frac{\# \text{parts}}{[s]} \text{ passing through ring}
\]
The particles in ring will scatter into curved ring of solid angle.
\[ d\Omega \equiv \text{ring} \, d\theta \, d\phi \]

\( d\Omega \) is "surface area" of red curved ring. This is what is measured in lab.

Particles scattered into \( d\Omega \).

Goal: find relation between in \( \Rightarrow d\omega \rightarrow d\Omega = \text{out} \)

\[ d\omega = D(\theta) \, d\Omega \]

\[ \frac{d\omega}{d\Omega} = D(\theta) \Rightarrow \]

Differential scattering cross section

Since units:
\[ [d\sigma] = [m^2] \]
\[ [d\Omega] = [1] \]
\[ [D(\theta)] = [m^2] = \text{Area} \]
\( \frac{d\sigma}{d\omega} = b d\theta d\phi \)

and \( d\omega = \sin \theta d\theta d\phi \)

\[ D(\theta) = \frac{\frac{d\sigma}{d\omega}}{\sin \theta d\theta d\phi} = \frac{b d\theta}{\sin \theta} \frac{d\phi}{d\theta} = \frac{b}{\sin \theta} \frac{d\phi}{d\theta} \]

Note that if the scattering potential is spherically symmetric, then \( d\phi \to \pi \)

\( d\sigma = 2\pi b d\theta d\phi \)

\( d\omega = 2\pi \sin \theta d\theta d\phi \)

We have entire incoming ring scattered into entire outgoing ring.

\[ 11.2 \text{ Hard sphere} \]

\( b = R \cos \theta / 2 \)

\[ \left| \frac{db}{d\theta} \right| = \left| -\frac{1}{2} R \sin \theta / 2 \right| = \frac{R}{2} \left| \sin \theta / 2 \right| \]

\[ D(\theta) = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{R \cos \theta / 2}{\sin \theta} \frac{R}{2} \left| \sin \theta / 2 \right| \]

But \( \sin 2\theta \equiv 2 \sin \theta \cos \theta \)

Take \( L4 = \theta \)

\[ D(\theta) = \frac{R^2}{2} \]

Independent of \( \theta \) for hard sphere!

Units of area, independent of \( b \)!
THINK OF THIS AS "PROBABILITY" THAT ALL PARTICLES IN INCOMING RING \( \Delta \Omega \) ARE SCATTERED INTO OUTGOING RING \( \Delta \Omega \)

\[
\Delta 11.7 \\
\Delta = S d\Omega \ D(e) = S d\Omega \ [d\tau \over d\tau]
\]

IS THE TOTAL CROSS-SECTION!

11.2 FOR HARD SPHERE

\[
\Delta = S D(e) \ d\Omega
\]

\[
= \frac{R^2}{4} S d\Omega
\]

\[
= \frac{R^2}{4} 4\pi
\]

\[
= \pi R^2
\]

THIS IS CROSS SECTIONAL AREA OF SPHERE LIKE "SHADOW"

ALL INCOMING PARTS IN SHADOW \( \pi R^2 \) ARE SCATTERED OUT OF BEAM
Typically the particle flux or current density

\[ \frac{1}{J} = \frac{\text{# particles}}{(s^2)[m^2]} \]

Is a vector to be integrated over the area of the inc. beam.

\[ I = S \cdot dA \cdot \overrightarrow{J} \]

If \( J \) is a constant and \( I \) to area

\[ I = S \cdot \overrightarrow{dA} \cdot \overrightarrow{J} = A \cdot \overrightarrow{J} \]

In this case \( \overrightarrow{J} = \overrightarrow{L} = \text{luminosity} \)

\[ \overrightarrow{L} = \frac{\text{#}}{(s^2)[m^2]} \]

Hence the \# particles passing through \( d\sigma \) per sec.

\[ d\sigma \rightarrow dN \]

\[ dN = \overrightarrow{L} \cdot d\sigma = \overrightarrow{L} (d\theta) d\Omega \]

\[ \left[ \frac{\#}{s} \right] = \left[ \frac{\#}{m^2 s} \right] \]

\[ D(\theta) = \frac{1}{\overrightarrow{L}} \frac{dN}{d\Omega} \]

If \( \overrightarrow{J} = \overrightarrow{J}(r, \theta) \) need to integrate instead.
Around 1900, Rutherford shot α particles from radioactive source at gold sheet 1 atom thick.

In 1900, two competing models of the atom:

- Jellium Model
  - Uniform + CHG.
  - Distribution with like + "jelly" with electrons stuck in like paisins
  - J.J. Thompson's model

- Orbital Model
  - + CHG at center
  - Electrons orbiting "smoralwitz" model

Orbital model had been rejected in 1900's since electron orbiting nucleus would be accelerating and radiate away energy and crash into nucleus in about 1 ns.

Jelly little scatter

Lots of scatter
Rutherford found some $x$'s bounced straight back at $\theta = 180^\circ$.

Impossible with jelly model.

So he resurrected orbital model (and proved it) winning Nobel prize.

Bohr stabilized orbit using quantization rule (Bohr's model).

Setup an incident & charge $q_1 = +ze$ and incident KE $T = \frac{1}{2}mv^2 = E$

Scatters off heavy, stationary, nucleous treated as point charge $q_2 = +79e$

For gold, only Coulomb interaction allowed.

\[ \text{Job is to find } D(\theta) \text{ what Rutherford measured in lab!} \]

From orbital mechanics we know

\[ \text{orbit is a hyperbola } \Rightarrow E_{\text{tot}} > 0 \]

Unbound orbit $\mathbf{r}(t) = \mathbf{r}(t) \hat{\mathbf{r}}$

The inst. velocity is $\mathbf{v}(t) = \dot{\mathbf{r}}(t) \hat{\mathbf{r}} + \dot{\mathbf{\xi}}(t) \hat{\mathbf{\xi}}$

Radial orbital tangential
\[ 1 \vec{v}^2 = \vec{v} \cdot \vec{v} = \vec{r}^2 + (\vec{r} \cdot \vec{\omega})^2 = \vec{r}^2 + \vec{r} \cdot \vec{\omega}^2 \]

\[ T = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m \left[ \vec{r}^2 + \vec{r} \cdot \vec{\omega}^2 \right] \]

Since Coulomb PE = \( V = k_0 \frac{Q \cdot Q}{r} \)

\[ k_0 = \frac{1}{4 \pi \varepsilon_0} \]

\[ E = T + V = \frac{1}{2} m \left[ \vec{r}^2 + \vec{r} \cdot \vec{\omega}^2 \right] + V = \text{const} \]

This is constant due to cons. E & R.

Now if momentum is \( \vec{l} = m \vec{r} \times \vec{v} \)

\[ = m [\vec{r} \times \vec{\omega}] \times [\vec{r} + \vec{r} \times \vec{\omega}] \]

\[ |\vec{l} \times \vec{\omega}| = 1 \]

\[ \vec{l} = 1 \vec{\omega} = m \vec{r}^2 \vec{\omega} = \text{const.} \]

\[ \vec{\omega} = \frac{\vec{l}}{mr^2} \]

\[ \frac{2}{m} \left[ E - V \right] = \vec{r}^2 + \vec{r} \cdot \vec{\omega}^2 \]

\[ = \vec{r}^2 + \vec{r} \left[ \frac{L^2}{m^2 r^4} \right] \]

\[ \frac{2}{m} \left[ E - V \right] = \vec{r}^2 + \frac{L^2}{m^2 r^2} \]

\( r \) is parameterized in t but we want \( r \) in terms of angle \( \alpha \)

Define \( u = \frac{1}{r} \)
\[ \dot{r} = \frac{dr}{dt} = \frac{dr}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt} \]

Since \( r = \frac{1}{u} = u^{-1} \)

\[ \frac{dr}{dt} = -u^{-2} = -\frac{1}{u^2} \]

Recall \( \frac{dx}{dt} = \dot{x} = \frac{L}{m r^2} \)

\[ \Rightarrow \dot{r} = \left[ -\frac{1}{u^2} \right] \left[ \frac{du}{dx} \right] \left[ \frac{L}{m r^2} \right] \]

\[ = \left[ -\frac{1}{u^2} \right] \left[ \frac{du}{dx} \right] \left[ \frac{L x}{m} \right] \]

\[ \Rightarrow \dot{r} = -\frac{L}{m} \frac{du}{dx} \]

Plug back into \( \dot{r} \)

\[ \frac{2}{m} \left[ E - V \right] = \frac{\dot{r}^2}{r^2} + \frac{L^2}{m r^2} \]

\[ = \frac{L^2}{m^2} \left( \frac{du}{dx} \right)^2 + \frac{L^2 u^2}{m^2} \]

\[ \Rightarrow \left( \frac{du}{dx} \right)^2 = \frac{2m}{L^2} \left( E - V \right) - u^2 \]

\[ \Rightarrow \frac{du}{dx} = \sqrt{\frac{2m}{L^2} \left( E - V \right) - u^2} \]

\[ \Rightarrow dx = \frac{du}{\sqrt{\frac{2m}{L^2} \left( E - V \right) - u^2}} = \sqrt{\frac{2m}{L^2} \left( E - V \right) - u^2} \]

\[ \therefore I(u) = \frac{2m}{L^2} \left( E - V \right) - u^2 \]
can't integrate yet since \( E \) and \( V \) depend on \( u \)!

It is \( t = 0 \Rightarrow r = \infty \), \( u = \frac{1}{r} = 0 \), \( \alpha = 0 \)

Let \( R_0 \) be point of closest approach.

\( r = \infty \)
\( u = 0 \)
\( \alpha = 0 \)

\( \Rightarrow \) \( u_0 = \frac{1}{R_0} = \text{const.} \)

At this point \( \alpha = \alpha_0 = \text{const.} \)

By symmetry same angle out

\( \Rightarrow \)

\( \alpha_0 + \alpha_0 + \theta = \frac{\pi}{2} \)

\( \Rightarrow \)

\( \theta = \pi - 2\alpha_0 = \text{const.} \)

From (××)

\[ \alpha_0 = \int_{0}^{u_0} \frac{du}{\sqrt{E(u)}} \]

Now need \( I(u) \)

\[ I(u) = \frac{2M}{L^2} \left[ E - V \right] - u^2 \]

\[ = \frac{2M}{L^2} \left[ E - k_0 \beta \gamma_2 u \right] - u^2 \]

\[ I(u) = \frac{2ME}{L^2} - \frac{2Mk_0 \beta \gamma_2 u}{L^2} - u^2 \]

\[ \Rightarrow \]

\[ (u_2 - u)(u - u_1) \]

where \( u_1 \) and \( u_2 \) are \( I \)'s two roots.
\[ \frac{du}{dx} = \sqrt{I(u)} \]

At point of closest approach \( u_0 \), \( u \) is not changing with \( x \)!

\[ \frac{du}{dx} \bigg|_{x=x_0} = 0 = \sqrt{I(u)} \approx \sqrt{I(u_0)} \]

\( \Rightarrow \ u_1, u_2, u_3 \text{ must be } u_0 \text{ the root!} \)

WLOG take \( u_2 > u_1 \) and \( u_0 = u_2 \)

\( \Rightarrow \ \theta = \pi - 2 \theta_0 \)

\[ \theta = \pi - 2 \int_0^{u_2} \frac{du}{\sqrt{(u_2-u)(u-u_1)}} \]

\[ \Rightarrow \ \theta = \pi + 2 \arcsin \left[ \frac{-u_2 + u_1 + u_2}{u_2 - u_1} \right] \bigg|_0^{u_2} \]

\[ \theta = -2 \arcsin \left[ \frac{u_1 + u_2}{u_2 - u_1} \right] \]

XXX so we now need to find root \( u_1, u_2 \) Q. Formula!

Now since \( E, L \) are const at \( t = -\infty \)

\[ \begin{align*} b &\rightarrow v_0^2 \\
|L| &= \left| m \frac{v_0}{b} \right| \Rightarrow m v_0 b \\
E &= \frac{1}{2} m v_0^2 \\
\Rightarrow \text{ initial velocity} \\
\text{combine to get (eliminate } v_0) \quad \frac{L^2}{2} &= 2 m b^2 E \Rightarrow \frac{2m}{L^2} = \frac{1}{b^2 E} \end{align*} \]
Recall \[ I(u) = \frac{2ME}{L^2} - \frac{2M}{L^2} k_0 \beta_0 \gamma \left( u - u^2 \right) \]

\[ \Rightarrow I(u) = \frac{1}{b^2} - \frac{1}{b^2} \left( \frac{k_0 \beta_0 \gamma}{E} \right) u - u^2 \]

\[ = \frac{1}{b^2} - \frac{\rho_0}{b^2} u - u^2 \]

\[ = \left[ \frac{1}{b^2} \right] - \left[ \frac{1}{b^2} \gamma \right] - \left[ \frac{1}{b^2} \right]^2 \]

or \[ -I(u) = u^2 + \frac{\rho_0}{b^2} u - \frac{1}{b^2} = 0 \]

Roots of \( I(u) \) are roots of \( -I(u) \)

Plug into Quad formula \( A = 1 \)

\[ B = \frac{\rho_0}{b^2} \]

\[ C = -\frac{1}{b^2} \]

\[ u_{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]

\[ u_{\pm} = \frac{-\rho_0/b^2 \pm \sqrt{\left(\frac{\rho_0/b^2}\right)^2 - 4 \cdot 1 \cdot \left(-\frac{1}{b^2}\right)}}{2} \]

Take \( u_2 = u_+ \) \( u_1 = u_- \)

\[ \Rightarrow \frac{u_1 + u_2}{u_2 - u_1} = \frac{-1}{\sqrt{1 + \left(\frac{2b}{\rho_0}\right)^2}} \]

Plug into \( \Theta \):

\[ \Theta = -2 \arcsin \left[ \frac{u_1 + u_2}{u_2 - u_1} \right] + 2 \arcsin \left[ \frac{1}{\sqrt{1 + \left(\frac{2b}{\rho_0}\right)^2}} \right] \]
\[ d = \frac{P_0}{2} \cot \frac{\theta}{2} = \frac{k_0 q_1 q_2}{E} \cot \frac{\theta}{2} \]

\[ D(\theta) = \left( \frac{b}{\sin \theta} \right) \left( \frac{\sin \theta}{d \theta} \right) = \frac{b}{\sin \theta} \left( \frac{P_0}{2} \right) \cot \frac{\theta}{2} \left( \frac{1}{2} \csc^2 \frac{\theta}{2} \right) \]

\[ = \frac{P_0}{2 \sin \theta} \cot \frac{\theta}{2} \left( \frac{P_0}{2} \right) \frac{1}{2} \frac{1}{\sin^2 \frac{\theta}{2}} \]

\[ = \frac{P_0^2}{16} \frac{1}{\sin^4 \theta/2} \frac{\cot \theta/2}{\sin^2 \theta/2} \frac{1}{\sin^2 \theta/2} \]

\[ = \frac{P_0^2}{16} \frac{1}{\sin^4 \theta/2} \]

\[ = \frac{k_0 q_1 q_2}{16 \pi E_0 E_n \sin^2 \theta/2} \]

\[ D(\theta) = \left( \frac{q_1 q_2}{16 \pi E_0 E_n \sin^2 \theta/2} \right) \]

\[ \text{Note: As } E \rightarrow \infty, D \rightarrow 0 \text{ little scattering prob if incident particle is too fast.} \]