PHYS 4112
H.W. No. 2 (SOLUTIONS)

① BOAS 10.3.2

$$\frac{du}{dx_i'} = \sum_j \frac{du}{dx_j} \cdot \frac{dx_j}{dx_i'} \quad \text{(by chain rule).}$$

$$= \frac{du}{dx_j} \cdot \frac{dx_j}{dx_i'} \quad (E \Sigma C).$$

Let $x_i' = a_{ij} x_j$

& $\cancel{x_j = b_{ji} x_i'}$ $x_j = b_{ji} x_i'$

But $a_{ij} = b_{ji}$ $\therefore$ $A^T = B$.

&

$$\frac{dx_j}{dx_i'} = b_{ji} = a_{ij}$$

So, $$\frac{du}{dx_i'} = \frac{du}{dx_j} \cdot a_{ij}$$

Note that $$\frac{du}{dx_i} = (\vec{\nabla} u)_i .$$

$$\Rightarrow (\nabla u)_i' = a_{ij} (\vec{\nabla} u)_j$$

So, $\vec{\nabla} u$ transforms like a Cartesian vector component-wise.

② BOAS 10.3.3.

$$T'_{\alpha\beta\gamma} = a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, T_{ijk}$$

Set $\alpha = \beta$ (contraction)

$$\Rightarrow T'_{\alpha\alpha\gamma} = \underbrace{a_{\alpha i}\, a_{\alpha j}}_{\Downarrow}\, a_{\gamma k}\, T_{ijk}$$

δ_{ij} ($\because$ product of 2 column vectors of a Xformation matrix ~~i.e.~~ which is orthogona

$$\Rightarrow T'_{\alpha\alpha\gamma} = a_{\gamma k}\, T_{iik}$$

So, T_{iik} is a rank-1 tensor.

③. BOAS 10.3.6.

$$T'_{\alpha\beta\gamma} = a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, T_{ijk}$$

$$S'_{\alpha\beta\gamma} = a_{\alpha l}\, a_{\beta m}\, a_{\gamma n}\, S_{lmn}$$

$$= a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, S_{ijk}$$ (indices can be reusec

$$\Rightarrow T'_{\alpha\beta\gamma} + S'_{\alpha\beta\gamma} = a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, (T_{ijk} + S_{ijk})$$

④ BOAS 10.4.1

$$\vec{L} = m\vec{r} \times (\vec{\omega} \times \vec{r})$$
$$= m\left[r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}\right]$$

$$\Rightarrow L_y = m\left[r^2\omega_y - (x\omega_x + y\omega_y + z\omega_z)y\right]$$
$$L_z = m\left[r^2\omega_z - (x\omega_x + y\omega_y + z\omega_z)z\right]$$

So, I_{ij}'s can be found to be:

$I_{yx} = -mxy$, $I_{yy} = m(x^2+z^2)$, $I_{yz} = -myz$

$I_{zx} = -mxz$, $I_{zy} = -myz$, $I_{zz} = m(x^2+y^2)$.

For an extended body,

$I_{yx} = -\int xy\,dm$, $I_{yy} = \int(x^2+\cancel{y}\,z^2)\,dm$, $I_{yz} = -\int yz\,dm$

$I_{zx} = -\int xz\,dm$, $I_{zy} = -\int yz\,dm$, $I_{zz} = \int(x^2+y^2)\,dm$

⑤ ~~The moment of inertia I~~

BOAS 10.4.4.

The moment of inertia tensor turns out to be:

$$I = \left(\frac{2}{15}\right)\begin{bmatrix} \pi & -1 & 0 \\ -1 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

The principle axes of inertia & their correspond moments can be found by diagonalizing I, as: P.T.O.

$(-1, 1, 0)$; $(0, 0, 1)$; $(1, 1, 0)$ – eigen vectors

$\dfrac{2(1+\pi)}{15}$; $\dfrac{2\pi}{15}$; $\dfrac{2}{15}(-1+\pi)$ – eigen values (correspond

⑥ BOAS 10.4.8

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v}\cdot\vec{v}$$

$$= \frac{1}{2} m \, \vec{v}\cdot(\vec{\omega}\times\vec{r})$$

$$= \frac{1}{2} m \, \vec{r}\cdot(\vec{v}\times\vec{\omega})$$

$$= \frac{1}{2} m \, \vec{\omega}\cdot(\vec{r}\times\vec{v})$$

$$= \frac{1}{2} \vec{\omega}\cdot(\vec{r}\times m\vec{v})$$

$$= \frac{1}{2} \vec{\omega}\cdot\vec{L}$$

$$= \frac{1}{2} \vec{\omega}\cdot\overleftrightarrow{I}\vec{\omega}$$

$$= \frac{1}{2} (\omega)^T \overleftrightarrow{I} \omega$$

if ω is written as a column vector, and ω^T, is ~~as~~ a row vector, & a row vector multipli by a column vector gives a scalar just as a dot product does!

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