The set $\{ p_\ell(x) \} = \left\{ \sqrt{\frac{2\ell+1}{2}}\, P_\ell \right\}$

are complete over $x \in [-1, 1]$ so

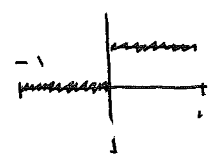
any function $f(x)$ may be expanded

in a series

$$\boxed{f(x) = \sum_{\ell=0}^{\infty} c_\ell\, p_\ell(x)}$$

where

$$\boxed{\begin{aligned} c_\ell &= \langle \ell | f \rangle \equiv \int_{-1}^{1} p_\ell^*(x)\, f(x)\, dx \\ &\equiv \sqrt{\frac{2\ell+1}{2}} \int_{-1}^{1} P_\ell(x)\, f(x)\, dx \end{aligned}}$$

Example $f(x) \equiv \left\{ \begin{array}{ll} 0 \; ; & -1 < x < 0 \\ 1 \; ; & 0 < x < 1 \end{array} \right\}$ 

STEP

$$c_\ell \equiv \langle \ell | f \rangle = \sqrt{\frac{2\ell+1}{2}} \int_{-1}^{1} P_\ell\, f\, dx$$

$$= \sqrt{\frac{2\ell+1}{2}} \int_{0}^{1} P_\ell(x)\, dx$$

We can use Eqs 2.8

$$c_0 = \sqrt{\frac{2\cdot 0+1}{2}} \int_0^1 P_0\,dx$$

$$= \frac{1}{\sqrt{2}} \int_0^1 1\,dx = \boxed{1/\sqrt{2}}$$

$$c_1 = \sqrt{\frac{2\cdot 1+1}{2}} \int_0^1 x\,dx = \sqrt{\frac{3}{2}}\ \frac{1}{2}x^2\Big|_0^1 = \boxed{\frac{1}{2}\frac{\sqrt{3}}{\sqrt{2}}}$$

$$c_2 = \sqrt{\frac{2\cdot 2+1}{2}} \int_0^1 \frac{1}{2}(3x^2-1)\,dx$$

$$= \sqrt{\frac{5}{2}}\left[\frac{1}{2}(x^3 - x)\right]_0^1$$

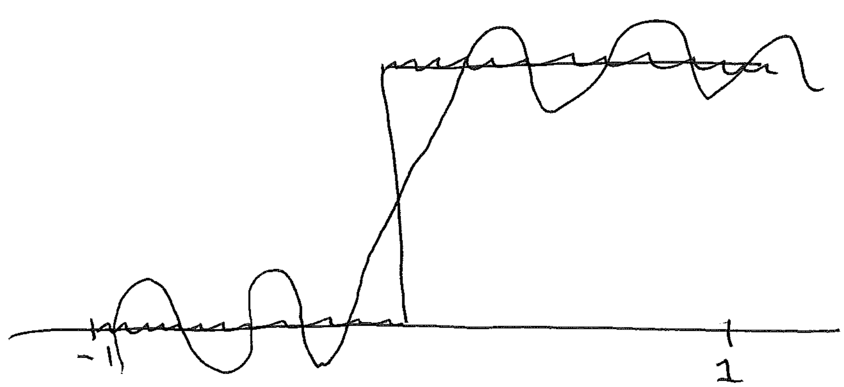
$$= \sqrt{\frac{5}{2}}\left[\frac{1}{2}(1-1) - 0\right] = \boxed{0}$$

Hence

$$f(x) = c_0 p_0 + c_1 p_1 + c_2 p_2$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2\cdot 0+1}{2}}\,P_0 + \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\,P_1 + 0\,P_2 + \cdots$$

$$= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) + 0\cdot P_2(x) + \left(-\frac{7}{16}\right)P_3 + 0\,P_4 + \cdots$$



Example

suppose we want to approximate

$$f(x) \cong ax^3 + bx^2 + cx + d$$

we demand

$$I \equiv \int_{-1}^{1} [f(x) - ax^3 - bx^2 - cx - d]^2 dx$$

is minimal

This is a least squares fit

The Legendre polynomials automatically give a, b, c, d!

Let $\quad p_\ell(x) = \sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)$

$$f(x) \cong c_0 p_0 + c_1 p_1 + c_2 p_2 + c_3 p_3$$

and

$$c_\ell = \langle \ell | f \rangle$$

$$= \int_{-1}^{1} p_\ell(x) f(x)\, dx$$

Then collect like powers of x.