$$\boxed{\int_{-1}^{1} dx\, P_\ell(x)\, P_\ell(x) = \int_{-1}^{1} dx\, P_\ell^2(x) = \frac{2}{2\ell+1}} \quad \text{Eq 8.1}$$

which we can write

$$\int_{-1}^{1} \left[\frac{P_\ell(x)}{\sqrt{\frac{2}{2\ell+1}}}\right]^* \left[\frac{P_\ell(x)}{\sqrt{\frac{2}{2\ell+1}}}\right] dx$$

$$= \int_{-1}^{1} \left[\sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)\right]^* \left[\sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)\right] = 1$$

which completes the proof that

$$\boxed{p_\ell(x) \equiv \sqrt{\frac{2\ell+1}{2}}\, P_\ell(x)} \quad \text{are orthonormal}$$

Proof From HW we have Eq. 5.86 Recursion

$$\boxed{\ell P_\ell(x) = x P_\ell'(x) - P_{\ell-1}'(x)}$$

$$\Rightarrow \boxed{\ell \int_{-1}^{1} dx\, P_\ell^2(x) = \int_{-1}^{1} x P_\ell P_\ell'\, dx - \underbrace{\int_{-1}^{1} P_\ell \underbrace{P_{\ell-1}'}_{m<\ell}\, dx}_{0} = \int_{-1}^{1} x P_\ell P_\ell'\, dx}$$

we now integrate by parts

note: $[P_\ell^2]' = 2 P_\ell P_\ell' \Rightarrow \frac{1}{2}[P_\ell^2]' = P_\ell P_\ell'$

$$l \int_{-1}^{1} P_l^2\,dx = \int_{-1}^{1} x \, P_l P_l' \, dx$$

	D	I
	x	$P_l P_l' \, dx$
$+$	1	$\frac{1}{2} P_l^2$
$-$		

$$= \frac{1}{2} x P_l^2 \Big|_{-1}^{1} - \frac{1}{2} \int_{-1}^{1} P_l^2 \, dx$$

but $\quad P_l(\pm 1) = \pm P_l(1) = \pm 1$

$$\Rightarrow \quad \frac{1}{2} x \, P_l^2(x) \Big|_{-1}^{1} = \frac{1}{2}(1)^2(\pm 1)^2 - \frac{1}{2}(-1)^2(\pm 1)^2 = 1$$

$$\Rightarrow \quad l \int_{-1}^{1} P_l^2(x)\,dx = 1 - \frac{1}{2}\int_{-1}^{1} P_l^2(x)\,dx$$

$$\Rightarrow \quad (l + 1/2) \int_{-1}^{1} P_l^2(x)\,dx = 1$$

$$\Rightarrow \quad \boxed{\int_{-1}^{1} P_l^2(x)\,dx = \frac{1}{l + 1/2} = \frac{2}{2l+1}} \quad \text{QED}$$