Hence $\forall \ell < m$

$$\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx$$

$$= \int_{-1}^{1} \sum_{j=0}^{\ell} A_j x^j\, P_m(x)\, dx$$

$$= \sum_j A_j \underbrace{\int_{-1}^{1} x^j\, P_m(x)\, dx}$$

$$= \sum_j A_j \cdot 0$$

$$= 0$$

Hence $\boxed{\int_{-1}^{1} P_\ell(x)\, P_m(x) \equiv 0 \quad \text{if } \ell < m}$

IF $\ell > m$ we just write

$$\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx$$

$$= \int_{-1}^{1} P_m(x)\, P_\ell(x)\, dx$$

$$= \int_{-1}^{1} \sum_{i=0}^{m} B_i x^i\, P_\ell(x)\, dx \qquad m < \ell$$

$$= \sum_{i=0}^{m} B_i \underbrace{\int_{-1}^{1} x^i\, P_\ell(x)\, dx}_{=0\ \forall i < \ell}$$

$$= 0$$

Hence we conclude

$$\boxed{\int_{-1}^{1} P_\ell(x)\, P_m(x) \equiv 0 \qquad \text{if} \quad \ell \neq m}$$

That is the set $\{P_\ell(x)\}_{\ell=0}^{\infty}$ is <u>orthogonal</u>

on $x \in [-1, 1]$