Recall in $3D = \mathbb{R}_3$ the vectors

$\{\vec{i}, \vec{j}, \vec{k}\}$ are a <u>complete</u> <u>ortho</u> normal

set Let $\{\vec{i}, \vec{j}, \vec{k}\} = \{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$

$= \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \equiv \{\hat{x}, \hat{y}, \hat{z}\}$

ORTHONORMAL $$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$$

complete

$\forall$ vectors $\vec{A} \in \mathbb{R}_3$ $\exists$ A_1, A_2, A_3 s.t.

$$\boxed{\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3}$$

These components may be found via

$$\begin{array}{|lcl|} A_1 & = & \vec{A} \cdot \hat{e}_1 \\ A_2 & = & \vec{A} \cdot \hat{e}_2 \\ A_3 & = & \vec{A} \cdot \hat{e}_3 \end{array} \quad \begin{array}{l} = \langle \vec{A} | 1 \rangle \\ = \langle \vec{A} | 2 \rangle \\ = \langle \vec{A} | 3 \rangle \end{array}$$

$\uparrow$

IN Dirac notation

$\hat{e}_i = | i \rangle$

$\hat{e}_i^* = \langle i |$ complex conjugate

and $\hat{e}_i^* \cdot \hat{e}_j = \hat{e}_i \cdot \hat{e}_j = \langle i | j \rangle = \delta_{ij}$

since $\hat{e}_i^* = \hat{e}_i$ are real.

IN ANALOGY

A set of functions $A_n(x)$ are a <u>complete</u> <u>ortho-normal</u> set on an interval $x \in [a,b]$ iff

$$\boxed{\int_a^b A_n^*(x)\, A_m(x)\, dx = \delta_{nm}}$$ ORTHO NORMAL

and

Any well behaved function $f(x)$ on the interval $x \in [a,b]$ may be written as

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n A_n(x)}$$

where we demand $\sum_n |c_n|^2 \equiv \sum_n c_n^* c_n \neq \infty$ for convergence. In Dirac notation:

$$A_n(x) = |n\rangle$$

$$A_n^*(x) = \langle n|$$

and

$$\boxed{\int_a^b A_n^*(x)\, A_m \equiv \langle n|m\rangle = \delta_{nm}}$$

The infinite dimensional vector space spanned by $\{A_n(x)\}_{n=0}^{\infty}$ is called a Hilbert space.

Example: From Fourier Analysis any well behaved function on the interval $x \in [-\pi, \pi]$ may be expanded in terms of the functions

$$A_n = |n\rangle = \frac{1}{\sqrt{2\pi}} e^{inx}$$

$$A_n^* = \langle n| = \left[\frac{1}{\sqrt{2\pi}} e^{inx}\right]^* = \frac{1}{\sqrt{2\pi}} e^{-inx}$$

where $\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=0}^{\infty}$

is a complete orthonormal set

ORTHONORMAL

$$\langle n|m\rangle = \int_{-\pi}^{\pi} \left[\frac{e^{inx}}{\sqrt{2\pi}}\right]^* \left[\frac{e^{imx}}{\sqrt{2\pi}}\right] dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{imx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx$$

$$= \frac{1}{2\pi} \begin{cases} 2\pi \ ; & m = n \\ \dfrac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{i(m-n)} & m \neq n \end{cases}$$

Recall $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} \equiv \sin\theta$

$$= \frac{1}{2\pi} \begin{cases} 2\pi & ; \quad m=n \\ \dfrac{2}{(m-n)} \dfrac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{2i} & ; \quad m \neq n \end{cases}$$

$$= \frac{1}{2\pi} \begin{cases} 2\pi & ; \quad m=n \\ \dfrac{2}{(m-n)} \sin(m-n)\pi & ; \quad m \neq n \end{cases}$$

$$= \begin{cases} 1 & ; \quad m=n \\ 0 & ; \quad m \neq n \end{cases} = \delta_{nm}$$

To find the coefficients $\exists\ c_n$ s.t.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} c_n |n\rangle$$

we compute

$$\langle m | f \rangle = \sum_{n=-\infty}^{\infty} c_n \overbrace{\langle m | n \rangle}^{\delta_{nm}}$$

$$= c_m$$

$$\equiv \int_{-\pi}^{\pi} \left[\frac{e^{imx}}{\sqrt{2\pi}} \right]^* f(x)\, dx$$

or dummy play $m \to n$

$$\boxed{c_n \equiv \langle n | f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)}$$