12.3 Leibnitz Rule

Recall Binomial Expansion

$$(x+y)^n = \frac{x^n}{0!}y^0 + \frac{n x^{n-1}}{1!}y^1 + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + x^0 y^n$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

where $\binom{n}{k} \equiv \frac{n!}{k!\,(n-k)!}$ is the binomial coefficient

We can also generate via pascals triangle

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & n=1 \\
 & & & & & & & 1 & 2 & 1 & & & & & & & n=2 \\
 & & & & & & 1 & 3 & & 3 & 1 & & & & & & n=3 \\
 & & & & & 1 & 4 & & 6 & & 4 & 1 & & & & & n=4 \\
 & & & & 1 & 5 & 10 & & & & 10 & 5 & 1 & & & & n=5 \\
 & & & 1 & 6 & 15 & & & 20 & & & 15 & 6 & 1 & & & n=6 \\
 & & 1 & 7 & 21 & 35 & & & & & & 35 & 21 & 7 & 1 & & n=7 \\
 & 1 & 8 & 28 & & & & & & & & & & & & & n=8
\end{array}$$

Hence

$$(x+y)^5 = 1 \cdot x^5 y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$$

This works for <u>differentiation</u>

$$(fg)^{(0)} = f^{(0)}g^{(0)} = fg$$

$$(fg)^{(1)} = (fg)' = f'g + fg' = f^{(1)}g^{(0)} + f^{(0)}g^{(1)}$$

$$(fg)^{(2)} = (fg)'' = f''g + 2f'g' + fg'' = f^{(2)}g^{(0)} + 2f^{(1)}g^{(1)} + f^{(0)}g^{(2)}$$

$$(fg)^{(3)} = (fg)''' = f'''g + 3f''g' + 3f'g'' = 1\,fg'''$$

$$\vdots$$

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g^{(0)} + \binom{n}{1} f^{(n-1)} g^{(1)} + \binom{n}{2} f^{(n-2)} g^{(2)} + \cdots + \binom{n}{n} f^{(0)}$$

In operator form

$$(fg)'' = \frac{d^2}{dx^2}(fg) = \frac{d^2f}{dx^2}g + 2\frac{df}{dx}\frac{dg}{dx} + f\frac{d^2g}{dx^2}$$

$$= \left[\left(\frac{d}{dx}\right)^2 f\right]g + 2\left[\left(\frac{d}{dx}\right)^1 f\right]\left[\left(\frac{d}{dx}\right)^1 g\right] + f\left[\left(\frac{d}{dx}\right)^2 g\right]$$

$$= \left[\left(\frac{d}{dx}\right)^2 f\right]\left[\left(\frac{d}{dx}\right)^0 g\right] + 2\left[\left(\frac{d}{dx}\right)^1 f\right]\left[\left(\frac{d}{dx}\right)^1 g\right] + \left[\left(\frac{d}{dx}\right)^0 f\right]\left[\left(\frac{d}{dx}\right)^1 g\right]$$

Hence

$$(fg)^{(n)} \equiv \left(\frac{d}{dx}\right)^n (fg)$$

$$= \binom{n}{0}\left[\left(\frac{d}{dx}\right)^n f\right]\left[\left(\frac{d}{dx}\right)^0 g\right] + \binom{n}{1}\left[\left(\frac{d}{dx}\right)^{n-1} f\right]\left[\left(\frac{d}{dx}\right)^1 g\right] + \cdots$$

$$+ \cdots \binom{n}{n}\left[\left(\frac{d}{dx}\right)^0 f\right]\left[\left(\frac{d}{dx}\right)^n g\right]$$

Ex 3.1 Find $\left(\frac{d}{dx}\right)^9 [x \sin x] \equiv (x \sin x)''''''''''$

$$= \binom{9}{0}\underbrace{(x)''''''}_{0}(\sin x) + \underbrace{\cdots +}_{0} \binom{9}{8}(x)'(\sin x)^{(8)} + \binom{9}{9} x(\sin x)^{(9)}$$

$$= 9(x)'(\sin x)^{(8)} + 1(x)(\sin x)^{(9)}$$

$$= \boxed{9 \sin x + x \cos x}$$

$\left(\frac{d}{dx}\right)^n \sin x = \pm \sin x$ n even; $\pm \cos x$ n odd

+ n even divis by 4

+ n odd n-1 divis by 4

Example

$$\left(\frac{d}{dx}\right)^5 \left(x^5 e^x \right)$$

note $\left(\frac{d}{dx}\right)^n e^x = e^x$ so factors

$$= \left[1(x^5)^{(5)} + 5(x^5)^{(4)} + 10(x^5)^{(3)} + 10(x^5)^{(2)} + 5(x^5)^{(1)} + 1\left(x^5\right)^{(0)} \right] e^x$$

$$= [\, 1 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0)$$
$$+ 5 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot x^1)$$
$$+ 10 \cdot (5 \cdot 4 \cdot 3 \cdot x^2)$$
$$+ 10 \cdot (5 \cdot 4 \cdot x^3)$$
$$+ 5 \cdot (5 \cdot x^4)$$
$$+ 1 \cdot (x^5) \,] e^x$$

Example

$$(x+1)^6$$

$$= 1 \cdot x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

By uniqueness of a solution to a Diffy Q

If $p_\ell(x)$ and $P_\ell(x)$ obey Legendre's DQ,

AND if $p_\ell(x)$ and $P_\ell(x)$ obey the same

two BC's $\quad p_\ell(1) = P_\ell(1) = \underbrace{1}_{\text{2nd}} \; \underbrace{\neq \infty}_{\text{1st}}$

Then $p_\ell(x) \equiv P_\ell(x) \;\; \forall x \quad$ and $\quad C \equiv 1$

To prove use hint in book Prob 12.4.2

$$(x^2-1)^\ell = [(x+1)(x-1)]^\ell = (x+1)^\ell (x-1)^\ell$$

$$\Rightarrow p_\ell(x) = \frac{1}{2^\ell \ell!} \partial^\ell \left[(x+1)^\ell (x-1)^\ell \right]$$

$$= \frac{1}{2^\ell \ell!} \sum_{m=0}^{\ell} \binom{\ell}{m} \partial^m (x+1)^\ell \, \partial^{\ell-m} (x-1)^\ell$$

By inspection all terms except $m=0$ will have an $(x-1)$ to some power and vanish when $x=1$ hence

$$p_\ell(x)\Big|_{x=1} = \frac{1}{2^\ell \ell!} \binom{\ell}{0} \partial^0 (x+1)^\ell \, \partial^\ell (x-1)^\ell \Big|_{x=1}$$

$$= \frac{1}{2^\ell \ell!} \cdot 1 \cdot (x+1)^\ell \cdot \ell \cdot (\ell-1) \cdot (\ell-2) \cdots (1)\, (x-1)^0 \Big|_{x=1}$$

$$= \frac{1}{2^\ell \ell!} \, 2^\ell \cdot \ell!$$

$$= 1 \qquad \text{QED}$$

$$\boxed{p_\ell(x) \equiv \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell = P_\ell(x)}$$

An easy way to generate any Legendre Polynomial is via the Rodriquez Formula

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell} \qquad \text{Eq. 4.1}$$

First we note $v \equiv (x^2-1)^\ell$ is a polynomial of degree 2ℓ. Hence if we take ℓ derivatives of it we get a polynomial of degree ℓ — that of P_ℓ. Let us abbreviate $\frac{d^\ell}{dx^\ell} \equiv \partial^\ell$, and

$$\partial^\ell v \equiv \frac{d^\ell v}{dx^\ell} = v^{(\ell)}$$ to simplify notation

Proof of Eq. 4.1

In our new notation we must prove

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\partial^\ell v = \frac{v^{(\ell)}}{2^\ell \ell!}$$

It is sufficient to prove that $v^{(\ell)}$ obeys the Legendre Diffy-Q Eq. 12.2.1

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

and that $$P_\ell(x) \equiv \left.\frac{v^{(\ell)}(x)}{2^\ell \ell!}\right|_{x=1} = 1$$

Note

$$\partial v = \frac{d}{dx}(x^2-1)^\ell = \ell(x^2-1)^{\ell-1}2x$$

$$\Rightarrow \quad (x^2-1)\partial v = \ell(x^2-1)^\ell 2x = \boxed{2\ell x v} \qquad \text{Eq. 12.4.3}$$

we now differentiate this $\ell+1$ times

$$RHS = \partial^{\ell+1}[2\ell x v] = \partial^{\ell+1}[(x^2-1)\partial v] = LHS$$

We now apply Leibnitz rule to both sides

$$LHS = \partial^{l+1}\left[(x^2-1)\,\partial v\right]$$

$$= \sum_{m=0}^{l+1} \binom{l+1}{m} \partial^{m}(x^2-1)\, \partial^{l+1-m}(\partial v)$$

Notice, like in example, that only $m = 0, 1, 2$ give non-zero terms since $\partial^3(x^2-1) = 0$

$$LHS = \binom{l+1}{0} \partial^0 (x^2-1)\, \partial^{l+1}(\partial v)$$

$$+ \binom{l+1}{1} \partial^1 (x^2-1)\, \partial^{l}(\partial v)$$

$$+ \binom{l+1}{2} \partial^2 (x^2-1)\, \partial^{l-1}(\partial v)$$

The binomials are

$$\binom{l+1}{0} \equiv \frac{(l+1)!}{0!\,(l+1-0)!} = 1$$

$$\binom{l+1}{1} \equiv \frac{(l+1)!}{1!\,(l+1-1)!} = l+1$$

$$\binom{l+1}{2} = \frac{(l+1)!}{2!\,(l+1-2)!} = \frac{(l+1)!}{2!\,(l-1)!} = \frac{(l+1)(l)}{2!} \equiv \frac{\lambda}{2!}$$

$$\boxed{\begin{aligned} LHS &= \left\{ (x^2-1)\,\partial^{l+2} v + (l+1)\cdot 2x \cdot \partial^{l+1} v + \frac{\lambda}{2!}\cdot 2\, \partial^{l} v \right\} \\ &= (x^2-1)\, v^{(l+2)} + (l+1)(2x)\, v^{(l+1)} + \frac{(l+1)\,l\cdot 2}{2!}\, v^{(l)} \end{aligned}}$$

which agrees with Eq. 4.4 Now we do the RHS

$$RHS = \partial^{l+1}\left[2 l x v\right]$$

$$= 2l\, \partial^{l+1}\left[x v\right]$$

$$= 2l \sum_{m=0}^{l+1} \binom{l+1}{m} \partial^{m}(x)\, \partial^{l+1-m}(v)$$

One again $\partial^0 x = x \quad \partial^1 x = 1 \quad \partial^2 x = 0$

and this time only two terms survive:

$$RHS = 2l\left[\binom{l+1}{0} x\,\partial^{l+1} v + \binom{l+1}{1}\partial^{l} v + 0 + 0 \dots\right]$$

$$= 2l\left[1 \cdot x\,\partial^{l+1} v + (l+1)\,\partial^{l} v\right]$$

$$\boxed{RHS = 2lx\, v^{(l+1)} + 2\lambda\, v^{(l)}}$$

Now set LHS = RHS

$$(x^2-1)\, v^{(l+2)} + \underbrace{(l+1)(2x)}_{2lx}\, v^{(l+1)} + \frac{\lambda}{2!}\, v^{(l)} = 2lx\, v^{(l+1)} + 2\lambda\, v^{(l)}$$

$\Rightarrow$

$$(x^2-1)\, v^{(l+2)} + 2x\, v^{(l+1)} - \lambda\, v^{(l)} = 0$$

$\Rightarrow$

$$\boxed{(1-x^2)\, v^{(l+2)} - 2x\, v^{(l+1)} + l(l+1)\, v^{(l)} = 0}$$

$$\Rightarrow \boxed{(1-x^2)\frac{\partial^2}{\partial x^2} v^{(l)} - 2x\frac{\partial}{\partial x} v^{(l)} + l(l+1)\, v^{(l)} = 0}$$

Hence $v^{(l)}$ satisfies Legendre Diffy-Q and so does

$$p_l(x) \equiv \frac{1}{2^l l!}\underbrace{\frac{d^l}{dx^l} v}_{v^{(l)}} = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

Hence $p_l(x) = C\,P_l(x)$ where C is a const.