12.19 Orthogonality of Bessel Functions

Consider $\sin x$ and $\cos x$

These are like ~~$J_0(x)$ and $J_1(x)$~~ J_p and Y_p

$\sin x = 0 \iff x = n\pi \; ; \; n = 0, \pm 1, \cdots$ roots

Rescale $\sin(n\pi x) = 0 \iff x = 0, \pm 1, \cdots$ roots

$\sin''(n\pi x) = -(n\pi)^2 \sin(n\pi x)$

$$\Rightarrow \boxed{\sin''(n\pi x) + (n\pi^2)\sin(n\pi x) \equiv 0}$$ scaled S.H.O.

$\sin(n\pi x)\big|_{x=0} = 0 \qquad \sin(n\pi x)\big|_{x=1,\dots} = 0$ B.C.

Consider $J_p(x)$ and $Y_p(x)$

like $\sin x$ $\cos x$

The roots of $J_p(x)$ are <u>not</u> integers but

we can count them $J_p(\alpha_p) = 0$, $J_p(\beta_p) = 0$, $\cdots$

where α_p, β_p, $\gamma_p \cdots$ are the roots of J_p

B.C. $J_p(x)\big|_{x=0} = 0, 1 \qquad J_p(\alpha_p x)\big|_{x=1} = 0$

Recall Fourier $\sin n\pi x$ are orthogonal

$$\int_0^1 \sin(n\pi x)\,\sin(m\pi x)\,dx$$

$\sin(n\pi x)$	$\sin(m\pi x)$
$n\pi \cos(n\pi x)$ (+)	$-\cos(m\pi x)/m\pi$
$-(n\pi)^2 \sin(n\pi x)$ (−) (+)	$-\sin(m\pi x)/(m\pi)^2$

$$= -\cancel{\sin(n\pi x)}^{\,0}\cos(m\pi x)/m\pi + n\pi\cos(n\pi x)\cancel{\sin(m\pi x)}^{\,0}/(m\pi)^2 \Big|_0^1$$

$$+ \left(\frac{n\pi}{m\pi}\right)^2 \int_0^1 \sin n\pi x\, \sin m\pi x\, dx$$

$$\Rightarrow \left(1 - \frac{n}{m}\right)\int_0^1 \sin n\pi x\,\sin m\pi x\,dx = 0$$

$$\Rightarrow \int_0^1 \sin n\pi x\,\sin m\pi x\,dx = 0 \quad \text{if } n \neq m$$

$$n = m \Rightarrow \int_0^1 [\sin n\pi x]^2 dx = \frac{1}{2}\int_0^1 [1 - \cos 2n\pi x]\, dx$$

$$= \frac{1}{2} \left[x - \cancel{\frac{\sin 2n\pi x}{2n\pi}}^{\,0} \right]_0^1 = 1/2$$

Hence $\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \frac{1}{2}\delta_{nm}$

and so $\{\sqrt{2}\sin(n\pi x)\}_{n=1}^{\infty}$ form an orthonormal set on $x \in [0,1]$ used in Fourier series analysis.

Now $x^2 J_p''(x) + x J_p'(x) + (x^2 - p^2) J_p(x) = 0$

We can rescale by roots α or β

$$\frac{d}{dx} J_p(x) = \frac{d\alpha x}{dx}\frac{d}{d(\alpha x)} J_p(\alpha x) = \alpha J_p'(\alpha x) \text{ etc.}$$

to get, using $x[x y']' = x y' + x^2 y''$,

$$\boxed{\begin{aligned} &x(x J_p'(\alpha x))' + [\alpha^2 x^2 - p^2] J_p(\alpha x) = 0 \\ &x(x J_p'(\beta x))' + [\beta^2 x^2 - p^2] J_p(\beta x) = 0 \end{aligned}}$$

where $\alpha = \alpha_p$ and $\beta = \beta_p$ are <u>any</u> two roots of $J_p(x) = 0$. To simplify notation

$u(x) \equiv J_p(\alpha x)$ and $v(x) \equiv J_p(\beta x)$

ORTHOGONALITY

$$\boxed{\int_0^1 x\, J_p(\alpha x)\, J_p(\beta x)\, dx = \frac{1}{2}\left[J_p'(\alpha)\right]^2 \delta_{\alpha\beta}}$$

Proof

$$x[x u]' + [\alpha^2 x^2 - p^2] u = 0 \quad * \qquad u \to \alpha$$

$$x[x v]' + [\beta x^2 - p^2] v = 0 \quad ** \qquad v \to \beta$$

we first prove ~~orthogonality for~~ $\alpha \neq \beta$

$$v(*) - u(**) = \boxed{v[xu']' - u[xv']' + (\alpha^2-\beta^2)xuv = 0} \quad ***$$

$$\Rightarrow \quad [vxu' - uxv']' + (\alpha^2-\beta^2)xuv = 0$$

since

$$v[xu']' = v[x'u' + xu'']$$

$$-u[xv']' = -u[x'v' + xv'']$$

$$= vu' + xvu'' - uv' - xuv''$$

and

$$[vxu' - uxv']' = \cancel{v'xu'} + vu' + vxu'' - \cancel{u'xv'} - uv'' - uxv'' =$$

Hence

$$\frac{d}{dx}[vxu' - uxv'] + (\alpha^2-\beta^2)xuv = 0$$

$$\Rightarrow \quad \overbrace{[vxu' - uxv']\Big|_0^1}^{=0 \text{ at } x=1 \text{ roots!}} + (\alpha^2-\beta^2)\int_0^1 xuv\,dx = 0 \quad ***$$

or

$$\Big[\underbrace{J(\beta x)\,x\,J'(\alpha x) - J(\alpha x)\,x\,J'(\beta x)}_{=0 \text{ at } x=0}\Big]\Big|_0^1 + (\alpha^2-\beta^2)\int_0^1 x\,J(\alpha x)J(\beta x)\,dx = 0$$

$$\Rightarrow \quad \underbrace{(\alpha^2-\beta^2)=0}_{\text{Nope! } \alpha\neq\beta} \quad \text{or} \quad \boxed{\int_0^1 x\,J_p(\alpha x)\,J_p(\beta x)\,dx = 0} \quad \text{QED}$$

IF $\alpha = \beta$ we write from ***

$$\lim_{\beta\to\alpha}\Big\{\underbrace{\big[J_p(\beta x)\,x\,J_p'(\alpha x) - J_p(\alpha x)\,x\,J_p'(\beta x)\big]}_{\text{~~as written before~~}}\Big|_0^1 + \lim_{\beta\to\alpha}(\alpha^2-\beta^2)\int_0^1 x\,J(\alpha x)J(\beta x)\,dx = 0$$

$$\Rightarrow \lim_{\beta\to\alpha} J_p(\beta x)\,x\,J_p'(\alpha x) - J_p(\alpha x)\,x\,J'(\beta x)\Big|_0^1 + \lim_{\beta\to\alpha}(\alpha^2-\beta^2)\int_0^1 x\,J(\alpha x)J(\beta x)\,dx = 0$$

STILL 0 aT $\Rightarrow 0$

$$\Rightarrow \underbrace{\lim_{\beta\to\alpha}\big[J_p(\beta)}_{\text{not root if } \beta\approx\alpha \quad \beta\neq\alpha} J_p'(\alpha) - \underset{\text{root}}{J_p(\alpha)}{}^{\to 0}\, J'(\beta)\big] + \lim_{\beta\to\alpha}(\alpha^2-\beta^2)\int_0^1 x\,J(\alpha x)J(\beta x) = 0$$

$\alpha = \beta$

Use hint problem 12.19.1 Let α = ROOT

but $\beta \approx \alpha$ WITH $\beta \neq$ ROOT and note Eq 19.7

(***) STILL HOLDS BUT $J(\beta) \neq 0$ and $J(\alpha) = 0$.

IT IS STILL OF FORM

$$\underbrace{\left[J(\beta x)\, x\, J'(\alpha x) - J(\alpha x)\, x\, J'(\beta x)\right]}_{I}\Bigg|_{x\to 0}^{x\to 1} + \underbrace{(\alpha^2-\beta^2)\int_0^1 x\, J(\alpha x)\, J_\beta(x)\, dx}_{II} = 0$$

BUT FIRST TERM IS NO LONGER ZERO.

$$\lim_{x\to 0} I_{BOT} = 0 \qquad \text{Bottom is still zero!}$$

$$\lim_{x\to 1} I_{TOP} = \lim_{x\to 1}\left[J(\beta x)\, x\, J'(\alpha x)\right] - \underbrace{\lim_{x\to 1}\left[\overbrace{J(\alpha x)}^{0}\ \overset{1}{x}\ \overbrace{J'(\beta x)}^{\text{const}}\right]}_{0}$$

but $\displaystyle \lim_{x\to 1} J'(\alpha x) = \lim_{x\to 1}\frac{d\,J(\alpha x)}{dx} = \lim_{x\to 1}\frac{d(\alpha x)}{dx}\frac{dJ(\alpha x)}{d(\alpha x)}$

$$= \lim_{x\to 1} \alpha \frac{d}{d(\alpha x)} J(\alpha x) = \alpha \lim_{x\to 1}\frac{d}{d(\alpha x)} J(\alpha x)$$

$$= \alpha \frac{d}{d\alpha} J(\alpha) = \alpha J'(\alpha)$$

Hence; $\displaystyle \lim_{\substack{x\to 1\ TOP\\ x\to 0\ BOT}} I = J(\beta)\,\alpha\, J'(\alpha)$

$$\Rightarrow \boxed{\frac{J(\beta)\,\alpha\, J'(\alpha)}{\beta^2-\alpha^2} = \int_0^1 x\, J(\alpha x)\, J(\beta x)\, dx}$$

Let $\beta \to \alpha$

$$\int_0^1 x\, J^2(\alpha x)\, dx = \lim_{\beta\to 0}\frac{\overbrace{J(\beta)}^{\to 0}\,\alpha\, J'(\alpha)}{\underbrace{\beta^2-\alpha^2}_{\to 0}} = \frac{0}{0}$$

LHOSPTIALS RULE

$$= \lim_{\beta\to 0}\frac{J'(\beta)\,\alpha\, J'(\alpha)}{2\beta} = \frac{1}{2}\left[J'(\alpha)\right]^2$$

$\boxed{\text{QED}}$

Recap

$$\int_0^1 x \, J_p[\alpha x] \, J_p[\beta x] \, dx = \frac{1}{2}\left[J_p'(\alpha)\right]^2 \delta_{\alpha\beta}$$

where α and β are roots of J_p.

There are two ways to look at this

I The functions

$$j_p(\alpha_p x) \equiv \sqrt{2}\left[J_p'(\alpha)\right]^{-1} \sqrt{x}\, J_p[\alpha x] \qquad \begin{matrix}\alpha \in \text{Roots} \\ \beta \in \text{Roots}\end{matrix}$$

are orthonormal on $x \in [0,1]$ that is

$$\int_0^1 j_p(\alpha_p x)\, j_p(\beta_p x)\, dx = \delta_{\alpha\beta}$$

As such we can expand well-behaved functions in terms of Bessel's Functions,

$$f(x) = \sum_\alpha C_\alpha \, j_p(\alpha x)$$

$$C_\alpha = \langle j_p(\alpha x) | f(x) \rangle = \int_0^1 j_p(\alpha x) f(x)\, dx$$

II The functions

$$g_p(\alpha_p x) \equiv \sqrt{2}\left[J_p'(\alpha)\right]^{-1} J_p[\alpha x]$$

are orthonormal w.r.t. the weight function $\rho(x) = x$ s.t.

$$\int_0^1 x \, g_p(\alpha_p x)\, g_p(\beta x)\, dx = \delta_{\alpha\beta}$$

$$f(x) = \sum_\alpha C_\alpha \, g_p(\alpha x)$$

$$C_\alpha = \langle f | g \rangle = \int_0^1 x \, f \, g \, dx$$