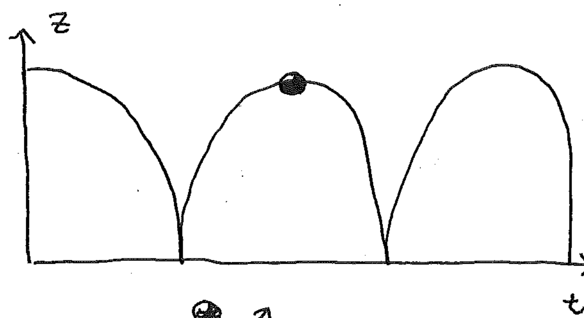
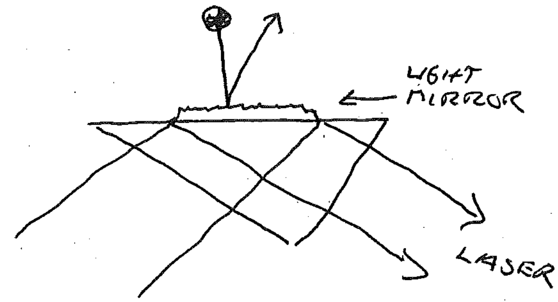
An atom is dropped from a trap onto a perfect mirror and bounces without friction. The mirror is made with an evanescent light field produced by total internal reflection and the wavelength is tuned to a repulsive "blue" side of a resonant transition in the atom.





If the thermal energy $kT \approx \frac{1}{2}mv^2 \approx mgz$ then atomic motion is quantized. The time independent Schrödinger Equation is

$$-\frac{\hbar^2}{2m}\Psi''(z) + V(z)\Psi(z) = E\Psi(z)$$

where z is the position of the atom above the mirror and $V(z) = mgz$ is the potential energy

We will apply two BC's

$$\Psi(z=\infty) \neq \infty \;/\; \Psi(\infty) = 0$$

$$\Psi(0) = 0$$

That is the wave function must vanish at zero and ∞.

We rewrite $$\Psi''(z) + \left[\frac{2mE}{\hbar^2} - \frac{2m^2 g}{\hbar^2} z\right]\Psi = 0$$

or $$\boxed{\Psi''(z) + [\epsilon - \gamma z]\Psi = 0}$$

Where $\varepsilon \equiv \frac{2mE}{\hbar^2}$ is scaled energy with units of $\frac{1}{[\text{Length}]^2}$ and $\gamma \equiv \frac{2m^2 g}{\hbar^2}$ has units $\frac{1}{L^3}$

We make the substitution

$$x = \frac{z\gamma - \varepsilon}{\gamma^{2/3}} \quad \text{which is dimensionless}$$

and note $\frac{d}{dz} = \frac{dx}{dz}\frac{d}{dx} = \gamma^{1/3}\frac{d}{dx}$

and $\frac{d^2}{dz^2} = \gamma^{2/3}\frac{d^2}{dx}$

and $\varepsilon - \gamma z = -\gamma^{2/3} x$

To get $\gamma^{2/3}\psi''(x) - \gamma^{2/3} x\,\psi(x) = 0$

$$\Rightarrow \boxed{\psi''(x) - x\psi(x) = 0}$$

which is just Airy's Diffy Q Eq. 17.8

The general solution is

$$\psi(x) = A\,Ai(x) + B\,Bi(x)$$

where Ai and Bi are the Airy Bessel Functions Eq 17.10

or

$$\boxed{\psi(z) = A\,Ai\left[\frac{z\gamma - \varepsilon}{\gamma^{2/3}}\right] + B\,Bi\left[\frac{z\gamma - \varepsilon}{\gamma^{2/3}}\right]}$$

To simplify our analysis we choose mass units such that $\gamma = 1$ and

$$\psi(z) = A\ Ai[z-\varepsilon] + B\ Bi[z-\varepsilon]$$

note in these units $V(z) = z$. For fixed ε we see that from notes

$$\lim_{x\to\infty} Ai[x] = 0 \qquad \lim_{x\to\infty} Bi[x] = \infty$$

and so Bi cannot be a solution as it blows up as $z \to \infty$. Hence apply the B.C. $\psi(\infty) = 0$ gives $B = 0$, const. integration.

And the acceptable solution is

$$\boxed{\psi(z) = A\ Ai[z-\varepsilon]}$$

At the other end we demand at mirror

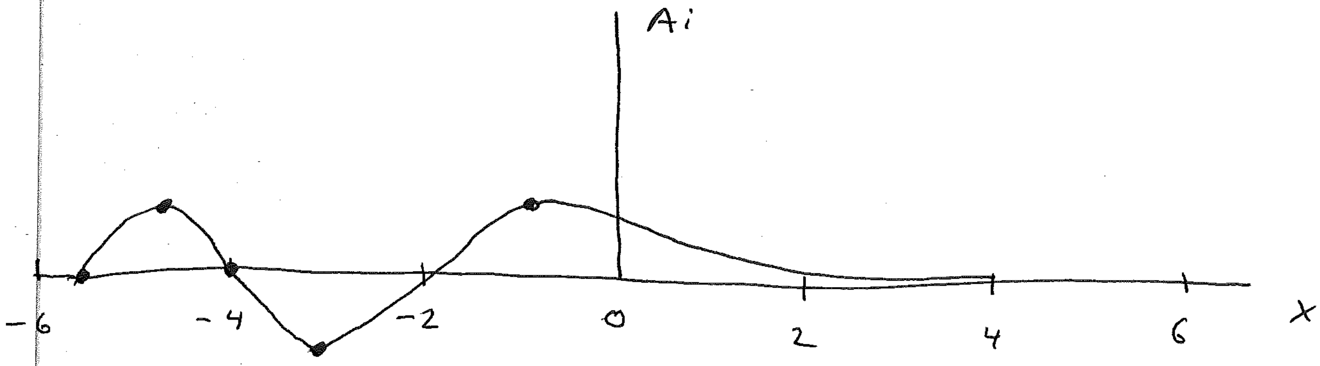
$$\psi(0) = 0 \quad \Rightarrow \quad \boxed{A\ Ai[-\varepsilon] \equiv 0}$$

since $A \neq 0$ to have a nontrivial solution we have a quantization condition

$$\boxed{Ai[-\varepsilon] = 0}$$

THAT IS ONLY ENERGYS S.T. $-\varepsilon$ IS A ROOT OF Ai are allowed! $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \varepsilon_3 \cdots\}$

Using mathematica we plot Ai[x]



From this we guess roots near $x = -2, -4, -5$ and use Root Finder to get the first 3 eigenenergies

$$\epsilon_1 \cong 2.34$$

$$\epsilon_2 \cong 4.09$$

$$\epsilon_3 \cong 5.52$$

The eigenfunctions are then

$$\boxed{\psi_i[z] = A_i \, Ai[z - \epsilon_i]} \qquad i = 1, 2, 3$$

To find the normalization constants we demand $\int_0^\infty dz \; \psi_i(z) = 1$

which we do numerically to get

$$A_1 = 1.43$$

$$A_2 = 1.25$$

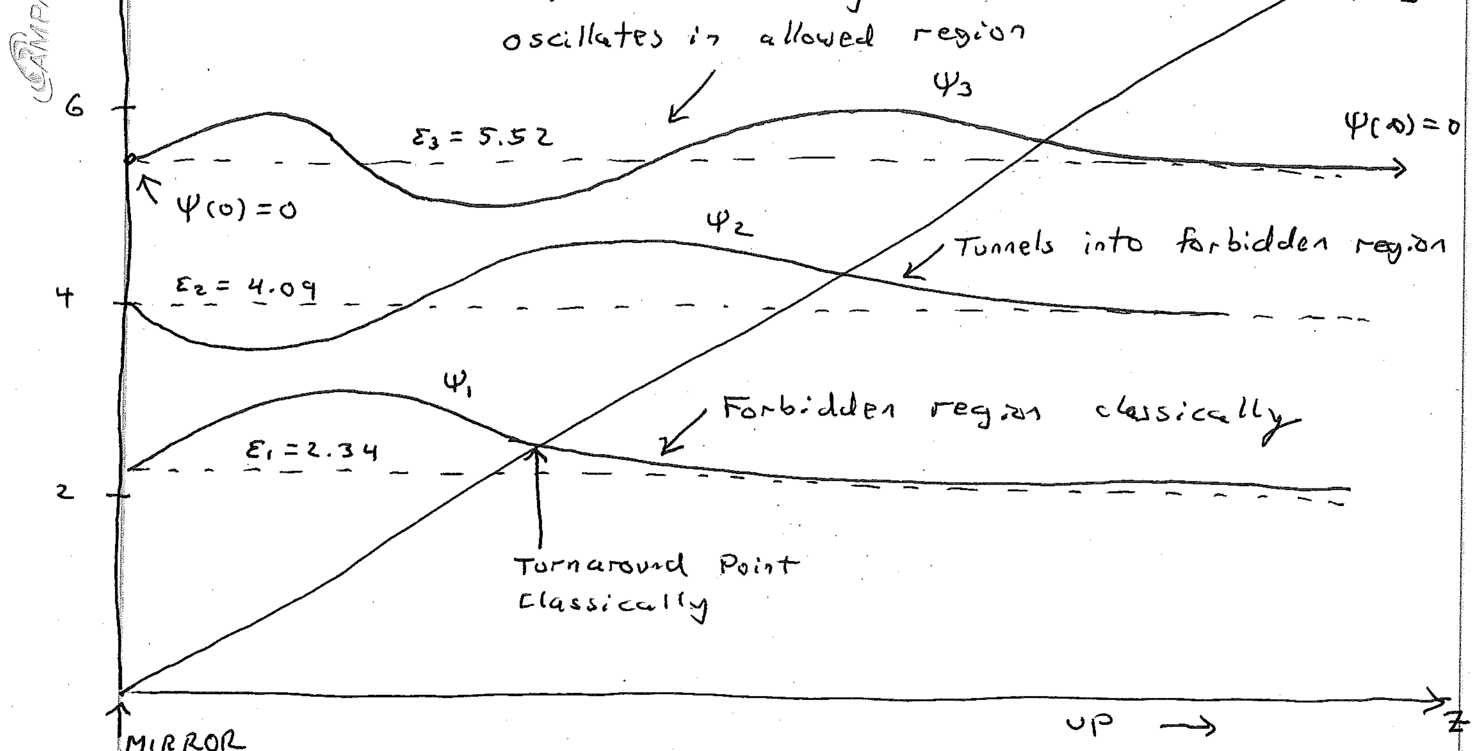
$$A_3 = 1.16$$

where $A_i \equiv \dfrac{1}{\sqrt{\int_0^\infty |\psi_i|^2 dz}}$

It is typical to plot

$$\boxed{\Psi_i(z) = \mathcal{E}_i + \psi_i(z)}$$

so the horizontal axis for each eigenfunction lies at its own energy level. Plotting with $V(z) = z$ we get



Classically the line $V = z$ corresponds to the point where $E = PE = mgh$ and all the energy is potential energy at the highest point, where ball turns around. Quantum ball can be found <u>above</u> the classical turnaround — it tunnels a bit into classically forbidden region. Also classical ball can have any energy whereas quantum ball has discrete energies $\mathcal{E}_i$.