Recall $y'' + k^2 y = 0$

has two Real independent solutions $\cos kx$ and $\sin kx$

we can construct complex solutions via

$$e^{\pm ikx} = \cos kx \pm i \sin kx$$

Similarly Bessel's Diffy-Q

$$x^2 y'' + x y' + [x^2 - p^2] y = 0$$

has two Real independent solutions

$$J_p(x) \underset{|x|>>1}{\cong} \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{p\pi}{2} - \frac{\pi}{4}\right]$$

$$Y_p(x) \underset{|x|>>1}{\cong} \sqrt{\frac{2}{\pi x}} \sin\left[x - \frac{p\pi}{2} - \frac{\pi}{4}\right]$$

we can construct complex solutions

$$\boxed{H_p^{(\pm)}(x) \equiv J_p(x) \pm i Y_p(x)}$$

Hankle

where $H_p^{(+)} = H_p^{(1)}$ and $H_p^{(2)} = H_p^{(-)}$

are the Hankel functions or Bessel's of the Third Kind.

Just as $\left\{ \begin{matrix} \cos kx \\ \sin kx \end{matrix} \right\}$ are good standing waves

and $e^{\pm ikx}$ are good propagating waves thus

J_p and Y_p are good standing

$H_p^{(\pm)}$ are good propagating.

We can combine the asymptotic expansions to get

$$H_p^{(\pm)}(x) \overset{|x|>>1}{\sim} J_p(x) \pm i\, Y_p(x)$$

$$= \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right) \pm i \sin\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$\boxed{H_p^{(\pm)}(x) \underset{|x|>>1}{\cong} \sqrt{\frac{2}{\pi x}} \exp\left[\pm i \left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)\right]}$$

Note since $H_p^{\pm}(x)$ contain $Y_p(x)$ we have

$$H_p^{\pm}(0) = -\infty \quad \text{like } Y_p(0)$$

Hyperbolic Bessel's

Recall $y'' + k^2 y = 0 \qquad k \in \mathbb{R}$

has solutions $e^{\pm ikx}$

which are propagating waves or

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2} \qquad \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

which are standing.

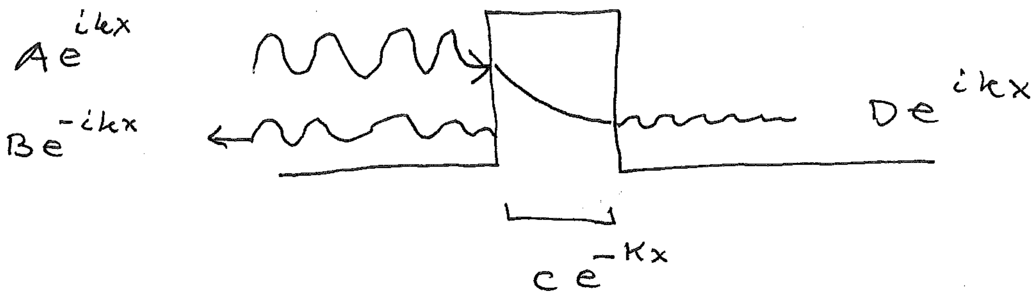
If we let $k = iK$ then Diffy Q is

$$y'' - K^2 y = 0$$

which has solutions $e^{\pm Kx}$ evanescent

or $\cosh(Kx) = \dfrac{e^{Kx} + e^{-Kx}}{2} \qquad \sinh(Kx) = \dfrac{e^{Kx} - e^{-Kx}}{2}$ hyperbolic

In Quantum Mechanics these are tunnelling solutions



Similar with Bessels. Note

$$\sin(ix) \equiv \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{+x}}{2i}$$

$$= i\left[\frac{e^{x} - e^{-x}}{2}\right] = i\sinh(x)$$

$$\Rightarrow \quad \sinh(x) = -i\sin(ix)$$
and $\cosh(x) = \cos(ix)$ $\Big]$ $x \in \mathbb{R}$

similarly if $x \in \mathbb{R}$

$$\boxed{\begin{aligned} I_p(x) &= i^{-p} J_p(ix) \\ K_p(x) &= \frac{\pi}{2} i^{p+1} H_p^{(+)}(ix) \end{aligned}}$$

are hyperbolic / evanescent / tunneling solutions.

If $p = m = 0, 1, 2, \cdots$ then

$$I_p(x) = I_m(x) \underset{|x|\gg 1}{\sim} i^{-m} J_m(ix)$$

$$\sim i^{-m}\sqrt{\frac{2}{\pi i x}} \cos\left[ix - \frac{m\pi}{2} - \frac{\pi}{4}\right]$$

some trig $\longrightarrow$ $\vdots$

$$\sim \frac{e^{x}}{\sqrt{2\pi x}} \qquad |x| \gg 1 \qquad p = 0, 1, 2, \ldots$$

$$\boxed{\begin{aligned} I_p(x) \underset{|x|\gg 1}{\sim} \frac{e^{x}}{\sqrt{2\pi x}} \\ K_p(x) \underset{|x|\gg 1}{\sim} \frac{e^{-x}}{\sqrt{2\pi x}} \end{aligned} \qquad p = m = 0, 1, 2, \ldots}$$

Spherical Bessels If $p = n + 1/2$; $n = 0, 1, 2, \cdots$ The solutions of Bessel's Diffy-Q can be written in terms of ordinary $\sin x$ and $\cos x$

$$j_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) = \overbrace{x^n \left[-\frac{1}{x}\frac{d}{dx}\right]^n \left(\frac{\sin x}{x}\right)}^{\text{Rodrigues Formula}}$$

$$y_n(x) \equiv \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) = -x^n \left[-\frac{1}{x}\frac{d}{dx}\right]^n \left[\frac{\cos x}{x}\right]$$

$$h_n^{(\pm)}(x) \equiv j_n(x) \pm i\, y_n(x)$$

are spherical Bessels' Functions. The arise in solution of scattering a plane wave off a spherical potential.

$$j_n(x) \underset{|x|\gg 1}{\sim} \frac{1}{x} \cos[x - (n+1)\pi/2]$$

$$y_n(x) \underset{|x|\gg 1}{\sim} \frac{1}{x} \sin[x - (n+1)\pi/2]$$

$$h_n^{\pm}(x) \underset{|x|\gg 1}{\sim} \frac{1}{x} \exp[\pm i\,(x - (n+1)\pi/2)]$$

Finally easy to show

$$j_n(x) \underset{|x|\ll 1}{\sim} \frac{2^n n!}{(2n+1)!} x^n \qquad \leftarrow \text{ Finite at } x=0$$

$$y_n(x) \underset{|x|\ll 1}{\sim} -\frac{(2n)!}{2^n n!} \frac{1}{x^{n+1}} \qquad \leftarrow \text{ Blow's up ! at } x=0$$

$$h_n^{\pm}(x) \underset{|x|\ll 1}{\sim} \mp \frac{i(2n)!}{2^n n!} \frac{1}{x^{n+1}} \qquad \leftarrow \text{ Blows up ! at } x=0$$

Kelvin Functions show up in heat flow and wave Eq with dissapation or loss

$$\boxed{x^2 y'' + xy' \qquad - i x^2 y = 0}$$

Kelvin's Diffy Q — Loss Term

Compare to Gen. Bess. Eq. 16.1

$$x^2 y'' + (1-2a) x y' + \left[(bc x^c)^2 + (a^2 - p^2 c^2) \right] y = 0$$

to get

$1 - 2a = 1 \Rightarrow \boxed{a = 0}$

$a^2 - p^2 c^2 = 0 \Rightarrow p^2 c^2 = 0$

$2c = 2 \Rightarrow \boxed{c = 1} \Rightarrow \boxed{p = 0}$

$b^2 = -i \Rightarrow b = \sqrt{-1}\,\sqrt{i} = i^{3/2}$

Hence Eq. 16.2 gives

$$y = Z_0(i^{3/2} x) \qquad Z_0 \in \{ J_0, Y_0, H_0^{\pm}, \dots$$

It is typical to define oscillating solutions

$$\boxed{J_0(i^{3/2}x) =: \mathrm{ber}(x) + i\,\mathrm{bei}(x)}$$

as well as exponentially decaying/growing solutions

$$\boxed{K_0(i^{1/2}x) =: \mathrm{ker}(x) + i\,\mathrm{kei}(x)}$$

That is

$$\mathrm{Ker}(x) \equiv \mathrm{Re}\{K_0(i^{1/2}x)\}$$

$$\mathrm{Kei}(x) \equiv \mathrm{Im}[K_0(i^{1/2}x)]$$

$$\mathrm{ber}(x) \equiv \mathrm{Re}[J_0[i^{3/2}x]]$$

$$\mathrm{bei}(x) \equiv \mathrm{Im}[J_0[i^{3/2}x]]$$

Airy's Diffy-Q Arises in E&M, optics, QM

$$\boxed{y'' - xy = 0}$$

write as

$$x^2y'' + 0\,y' + [(-x^3) + 0]y = 0$$

compare to Eq. 16.1

$$x^2y'' + (1-2a)xy' + [(bcx^c)^2 + a^2 - p^2c^2]y = 0$$

$$\Rightarrow \quad 1 - 2a = 0 \Rightarrow \boxed{a = 1/2}$$

$$2c = 3 \Rightarrow \boxed{c = 3/2}$$

$$(1/2)^2 - p^2(3/2)^2 = 0 \Rightarrow \frac{1}{4} = p^2\, 9/4 \Rightarrow \boxed{p = 1/3}$$

$$(bc)^2 = -1$$

$$bc = i$$

$$\boxed{b = \frac{2}{3} i}$$

Hence solutions are of form

$$x^a Z_p[bx^c] = \boxed{\sqrt{x}\, Z_{1/3}[\tfrac{2}{3} i\, x^{3/2}]}$$

where $Z_{1/3} \in \{ J_{1/3},\ Y_{1/3},\ H^{\pm}_{1/3},\ I_{1/3},\ K_{1/3} \}$

a particular "canned" linear combination gives Airy functions

$$\boxed{\begin{aligned} Ai[x] &= \frac{1}{\pi}\sqrt{\frac{x}{3}}\, K_{1/3}\left(\frac{2}{3}x^{3/2}\right) \\ Bi[x] &= \sqrt{\frac{x}{3}}\left[I_{-1/3}(2/3\, x^{3/2}) + I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] \end{aligned}}$$

Note that by using $|x| >> 1$ expressions for K and I we get

$$Ai[x] \underset{|x|>>1}{\sim} \frac{1}{\pi}\sqrt{\frac{x}{3}}\, \frac{e^{-\frac{2}{3}x^{3/2}}}{\sqrt{2\pi x}} = \boxed{\frac{1}{\sqrt{6\pi^3}}\, e^{-\frac{2}{3}x^{3/2}}} \xrightarrow[x\to\infty]{} 0$$

$$Bi[x] \underset{|x|>>1}{\sim} \sqrt{\frac{x}{3}} \cdot 2\, e^{+\frac{2}{3}x^{3/2}} \frac{1}{\sqrt{2\pi x}}$$

$$= \boxed{\sqrt{\frac{2}{3\pi}}\, e^{+\frac{2}{3}x^{3/2}}} \xrightarrow[x\to\infty]{} \infty$$

and so Bi blows up as $x \to \infty$, but Ai goes rapidly to zero as $x \to \infty$.