The original Bessel's Diffy-Q

$$\boxed{x^2 y'' + x y' + (x^2 - p^2) y = 0}$$

Typically $x = kr$ where $k = \frac{2\pi}{\lambda}$

for solution to wave equation radiall part.

Since $y'(x) = \frac{dy}{dx} = \frac{dy}{dr}\frac{dr}{dx} = \frac{1}{k}\frac{dy}{dr}$

$y''(x) = \frac{d^2y}{dx^2} = \frac{d^2y}{dr^2}\frac{dr}{dx}\frac{dr}{dx} = \frac{1}{k^2}\frac{dy^2}{dr^2}$

Then The diffy Q becomes

$$\frac{[\cancel{k}r]^2}{\cancel{k^2}}\frac{d^2y}{dr^2} + \frac{[\cancel{k}r]}{\cancel{k}}\frac{dy}{dr} + [(kr)^2 - p^2]y = 0$$

$$\Rightarrow \boxed{\frac{dy^2}{dr^2} + \frac{dy}{dr} + [k^2r^2 - p^2]\,y(r) = 0}$$

which has solutions

$$y_k(r) = J_p(kr) \text{ or } Y_p(kr)$$

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$$\boxed{x^2 y''(x) + x(1-2a)y'(x) + [(bcx^c)^2 + (a^2 - p^2c^2)]y(x) = 0}$$

has solution

$$\boxed{y(x) = x^a Z_p(bx^c)} \qquad \text{where} \quad Z_p = \begin{cases} Y_p \\ J_p \end{cases}$$

proof Let $y(x) = x^a u(z)$

where $z \equiv bx^c$

Hence $\dfrac{du}{dx} = \dfrac{dz}{dx}\dfrac{du}{dz} = bcx^{c-1}\dfrac{du}{dz} = bcx^{c-1}u'(z)$

Thus

$$y'(x) = \frac{d}{dx}[x^a u] = ax^{a-1}u + x^a(bcx^{c-1})u'(z)$$

$$\boxed{xy' = ax^a u + bcx^a x^c u'}$$

$$\begin{aligned} y''(x) = \frac{d}{dx}y'(x) &= a(a-1)x^{a-2}u \\ &+ ax^{a-1}\frac{du}{dx} \\ &+ bc(a+c-1)x^{a+c-2}u'(z) \\ &+ bcx^{a+c-1}\frac{du'(z)}{dx} \end{aligned}$$

$$\begin{aligned} &= a(a-1)x^{a-2}u(z) \\ &+ ax^{a-1}bcx^{c-1}u'(z) \\ &+ bc(a+c-1)x^{a+c-2}u'(z) \\ &+ (bc)^2 x^{a+c-1}x^{c-1}u''(z) \end{aligned}$$

$$\Rightarrow x^2y'' = a(a-1)x^a u$$
$$+ abc\, x^{a+c} u'$$
$$+ bc(a+c-1)x^{a+c} u'$$
$$+ (bc)^2 x^{a+2c} u''$$

Putting into $\circledast$ and collect like terms

$$(bc)^2 x^{a+2c} u'' + [abc + bc(a+c-1)]x^{a+c}u' + a(a-1)x^a u$$
$$+ (1-2a)[a x^a u + bc\, x^{a+c} u']$$
$$+ [(bc\,x^c)^2 + (a^2 - p^2c^2)]x^a u \qquad \text{Recall } bx^c = z$$

$$= c^2 x^a z^2 u'' + [ac + c(a+c-1)]x^a z u' + a(a-1)x^a u$$
$$+ (1-2a)[a x^a u + c x^a z u']$$
$$+ [c^2z^2 + (a^2 - p^2c^2)]x^a u$$

$$= c^2 x^a z^2 u''$$
$$+ [ac + c(a+c-1) + (1-2a)c]x^a z u'$$
$$+ [a(a-1) + (1-2a)a + (a^2 - p^2c^2) + c^2z^2]x^a u$$

$$= c^2 x^a z^2 u''$$
$$+ [\cancel{ac} + \cancel{ac} + c^2 - \cancel{c} + \cancel{c} - \cancel{2ac}]x^a z u'$$
$$+ [\cancel{a^2} - \cancel{a} + \cancel{a} - \cancel{2a^2} + \cancel{a^2} - p^2c^2 + c^2z^2]x^a u$$

$$= c^2 x^a z^2 u''$$
$$+ c^2 x^a z u'$$
$$+ [c^2z^2 - p^2c^2]x^a u$$

$$= c^2 x^a \underbrace{\left[z^2 u'' + z u' + (z^2 - p^2) u \right]}_{\text{This is regular Bessel's Diffy-Q}}$$

Hence the whole thing $= 0$ if

$$u(z) = J_p(z) \quad \text{or} \quad Y_p(z) \equiv Z_p(z)$$

$$\Rightarrow \quad y = x^a u(z) = x^a Z_p(z)$$

$$= x^a Z_p(b x^c) \qquad \text{QED}$$

Linear Potential

Example solve $y'' + 9xy = 0$ Airy's Diffy Q

$$\Rightarrow \quad x^2 y'' + 9x^3 y = 0$$

$$\Rightarrow \quad x^2 y'' + x(1-2a) y' + \left[b^2 c^2 x^{2c} + (a^2 - p^2 c^2) \right] y = 0$$

compare then $1 - 2a = 0$ no y' term

$$\Rightarrow \boxed{a = 1/2}$$

$$(a^2 - p^2 c^2) = 0 \Rightarrow 1/4 - p^2 c^2 = 0 \Rightarrow \boxed{\tfrac{1}{2} = pc}$$

and $b^2 c^2 = 9 \Rightarrow \boxed{bc = 3}$

and $2c = 3 \Rightarrow \boxed{c = 3/2}$

$$\Rightarrow \quad \tfrac{1}{2} = p \cdot 3/2 \Rightarrow \boxed{p = 1/3}$$

$$\Rightarrow \quad b \cdot 3/2 = 3 \Rightarrow \boxed{b = 2}$$

Read off a solution

$$y(x) = x^a Z_p(bx^c)$$

$$= \sqrt{x}\, Z_{1/3}[2x^{3/2}]$$

and hence general solution is

$$\boxed{y(x) = A\sqrt{x}\, J_{1/3}[2x^{3/2}] + B\sqrt{x}\, Y_{1/3}[2x^{3/2}]}$$