Since Bessel's Diffy Q is second order there are <u>two</u> linearly independent solutions for $p > 0$

$J_p(x)$ First solution

$J_{-p}(x)$ second solution

Hence general solution is

$$\boxed{y(x) = A\,J_p(x) + B\,J_{-p}(x)} \qquad \begin{matrix} p > 0 \\ p \neq 1, 2, 3 \cdots \end{matrix}$$

We can generate a formula for $-p < 0$ by just replacing $p \to -p$ in the series solution

$p > 0$
$p \neq 1, 2, \cdots$

$$J_p = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}$$

$$J_{-p} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n-p}$$

so long as $\begin{matrix} p > 0 \\ p \neq 1, 2, 3 \end{matrix}$ then

$$\boxed{J_{-p} \underset{|x| \ll 1}{\sim} \frac{1}{\Gamma(1-p)} \left(\frac{x}{2}\right)^{-p}}$$

which is singular at $x = 0$.

If $p \neq 0, 1, 2, 3$ we have a problem in that J_{-p} is not an independent solution.

$$p = 0, 1, 2, \cdots = m \Rightarrow \boxed{J_{-m}(x) = (-1)^m J_m(x)}$$

which differ by a constant and so are linearly dependent!

Proof $$J_{-m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-m)} \left(\frac{x}{2}\right)^{2n-m}$$

Recall $\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \cdots = \infty$

Hence $1/\Gamma(0) = 1/\Gamma(-1) = 1/\Gamma(-2) = \cdots = 0$

So all terms in the series from $n+1-m \leq 0$ are zero and so series starts at $n+1-m = 1 \ni 0!$

or $n: m, m+1, \cdots \infty$

$$\Rightarrow J_{-m} = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,(n-m)!} \left(\frac{x}{2}\right)^{2n-m}$$

let $l = n-m$
$n: m \to \infty$
$l: 0 \to \infty$
$n = l+m$

$$= \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{(l+m)!\,l!} \left(\frac{x}{2}\right)^{2l+m}$$

$$= (-1)^m \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+m)!} \left(\frac{x}{2}\right)^{2l+m}$$

$$= (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2n+m}$$

$$\equiv (-1)^m J_n(x)$$

Hence when $p = m = 0, 1, 2, 3 \dots$

we need a second solution that is linearly independent

Recall $y'' + y = 0 \Rightarrow y = A\cos x + B\sin x$

$$= \tilde{A} e^{ix} + \tilde{B} e^{-ix}$$

where $e^{\pm ix} = \cos x \pm i \sin x$

and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

So we can construct a new linearly independent solution

$$\boxed{Y_p(x) \equiv N_p(x) = \frac{\cos(\pi p) J_p - J_{-p}}{\sin \pi p}}$$

This has the form $\frac{0}{0}$ if $p = 0, 1, 2, 3 \equiv m$

and so we define

$$Y_m = N_m = \lim_{p \to m} \frac{\cos(\pi p) J_p - J_{-p}}{\sin(\pi p)} \qquad m = 0, 1, 2, \dots$$

This is Bessel's Function of the <u>second</u> kind

Also called Neuman or Weber fcn.

For $m = 0, 1, 2, \cdots$ we apply L'Hospital's Rule to get

$m = 0, 1, 2, \cdots$

$$\boxed{Y_m(x) = \frac{2}{\pi}\left[\ln(x/2) + \gamma\right] J_m(x) - \frac{1}{\pi}\sum_{n=0}^{m-1} \frac{(m-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-m}}$$

where $\gamma \cong 0.577$ is Euler's Constant

$$\equiv \lim_{n\to\infty}\left[\underbrace{\sum_{m=1}^{n}\frac{1}{m}}_{\text{Diverges}} - \ln n\right]$$

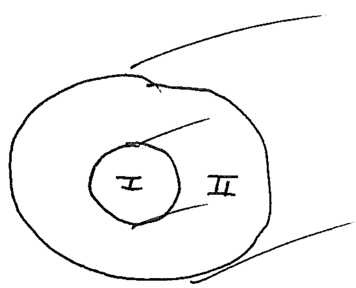
Hence $Y_m(0) = \infty \quad \forall\ m = 0, 1, 2, \cdots$

since $\ln(0) = -\infty$

Typically $Y_m(x)$ is discarded as a possible physical solution at the origin.

Example Solve wave equation for a dielectric waveguide

$$\nabla^2\psi + k^2\psi = 0$$



$\psi_I = A\, J_m(x)$ General solution $\neq \infty$ at $r=0$

$\psi_{II} = C\, J_m(x) + D\, Y_m(x)$ General solution $\neq \infty$ at $r > 0$

$$\boxed{Y_m(x) \underset{|x|\gg 1}{\sim} \sqrt{\frac{2}{\pi x}}\,\sin\left[x - \frac{m\pi}{2} - \frac{\pi}{4}\right]}$$ Asymptotic