Previously we considered series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

For well-behaved functions at the origin

However in physics some functions are not well behaved

$$V(r) = \frac{e^2}{r}$$

has a simple pole singularity at $r=0$.

we want to handle such functions as solutions to Diffy-Q's

$$y(x) = \frac{\cos x}{x^2} = \frac{1}{x^2}\left[1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right]$$

$$= \Big[\underbrace{\frac{1}{x^2}}_{\text{singular}} \underbrace{- \frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots}_{\text{Well-Behaved}}$$

or

$$y = \sqrt{x}\,\sin x = \sqrt{x}\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right.$$

has no derivaties at $x=0$

Both can in the Frobnius Form

$$\boxed{y(x) = x^s \sum_{n=0}^{\infty} a_n x^n}$$

$s \in \mathbb{R}$ $s=-2$ $s = 1/2$ etc.

Example

$$\boxed{x^2 y'' + 4xy' + (x^2+2)y = 0} \qquad \text{Eq. } 11.2$$

Note that $|x| \ll 1$ we can neglect $x^2 \lll 1$ to get

$$4xy' + 2y \cong 0 \qquad |x| \ll 1$$

$$2xy' \cong y$$

$$\Rightarrow \quad 2x\frac{dy}{dx} \cong y \quad \Rightarrow \quad 2\frac{dy}{y} = \frac{1}{x}\,dx$$

$$\Rightarrow \quad e^{2\ln y} \cong e^{\ln x + c}$$

$$\Rightarrow \quad y^2 = e^c x$$

$$\Rightarrow \quad \boxed{y \cong A\sqrt{x} \qquad |x| \ll 1}$$

This tells us that $y'(0)$ is not defined at $x=0$ and so we must use generalized power series

$$\boxed{\begin{aligned} y(x) &= x^s \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+s} \\ y'(x) &= \sum_{n=0}^{\infty} (n+s)\, a_n x^{n+s-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n x^{n+s-2} \end{aligned}}$$

we insert into Eq. 11.2

$$x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n\, x^{n+s-2}$$

$$+ 4x \sum_{n=0}^{\infty} (n+s)\, a_n\, x^{n+s-1}$$

$$+ x^2 \sum_{n=0}^{\infty} a_n x^{n+s} + 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n\, x^{n+s}$$

$$+ 4 \sum_{n=0}^{\infty} (n+s)\, a_n\, x^{n+s}$$

$$+ \boxed{\sum_{n=0}^{\infty} a_n\, x^{n+s+2}}^{\,*} + 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0$$

only * runs at wrong speed and needs to be resummed

$$* = \sum_{n=0}^{\infty} a_n x^{n+s+2} \qquad m = n+2 : 2 \to \infty$$
$$n = m-2 : 0 \to \infty$$

$$= \sum_{m=2}^{\infty} a_{m-2}\, x^{m+s} \qquad m \to n \quad \text{dummy}$$

$$= \sum_{n=2}^{\infty} a_{n-2}\, x^{n+s}$$

Putting this back in

$$\Rightarrow \boxed{\begin{aligned} &\sum_{n=0}^{\infty} (n+s)(n+s-1)\, a_n\, x^{n+s} \\ &+ \sum_{n=0}^{\infty} 4(n+s)\, a_n\, x^{n+s} \\ &+ \sum_{n=2}^{\infty} a_{n-2}\, x^{n+s} + \sum_{n=0}^{\infty} 2 a_n x^{n+s} = 0 \end{aligned}} \quad **$$

We now can read off coefficients of x^{n+s}

but first note $n=0$ and $n=1$ are "special"

$n=0$: $(0+s)(0+s-1)a_0$

$+ 4(0+s)a_0$

$+ 0 + 2a_0 = 0$

$$\Rightarrow \boxed{[s(s-1)+4s+2]a_0 = 0}$$

For a general solution $a_0 \neq 0$ so

$$s(s-1)+4s+2 \equiv 0$$
$$\Rightarrow s^2 - s + 4s + 2 = 0$$
$$\Rightarrow s^2 + 3s + 2 = 0$$
$$\Rightarrow (s+1)(s+2) = 0$$

$$\Rightarrow \boxed{s = -1 \text{ or } s = -2}$$

These correspond to two linearly independent solutions! Like $y'' + y = 0 \Rightarrow y = A\sin x + B\cos x$

now let us take the case $s = -1$ First

This will generate its own series. IN ** we get

$n = 1$, $s = -1$: $(1-1)(1-1-1)a_1$

$+ 4(1-1)a_1$

$+ 0 + 2a_1 = 0$

$$\Rightarrow \boxed{a_1 = 0}$$

$n > 1$, $s = -1$: $(n-1)(n-1-1)a_n$

$+ 4(n-1)a_n$

$+ a_{n-2} + 2a_n = 0$

$$\Rightarrow \boxed{\underbrace{(n-1)(n-2)a_n + 4(n-1)a_n}_{\text{FACTOR}} + 2a_n = -a_{n-2}}$$

$$\Rightarrow \quad [(n-1)(n-2+4)+2]\, a_n = -a_{n-2}$$

$$\Rightarrow \quad [(n-1)(n+2)+2]\, a_n = -a_{n-2}$$

$$\Rightarrow \quad [n^2+2n-n-\cancel{2}+\cancel{2}]\, a_n = -a_{n-2}$$

$$\Rightarrow \quad [n(n+1)]\, a_n = -a_{n-2}$$

$$\Rightarrow \quad \begin{matrix} s=-1 \\ n \geq 2 \end{matrix} \qquad \boxed{a_n = -\frac{a_{n-2}}{n(n+1)}} \qquad \text{Eq. } 11.4.5$$

we may set $m = n-2 \to n$

$$\begin{matrix} s=-1 \\ n \geq 0 \end{matrix} \qquad \boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+3)}}$$

This is our recursion relation! Since $a_1 = 0$ this gives

$$a_3 = -\frac{1}{(1+2)(1+3)} \cdot a_1 = 0$$

$$a_5 = -\frac{1}{(5+2)(5+3)} \cdot a_3 = 0$$

$$\vdots$$

$$a_{2m+1} \equiv 0 \qquad \forall \quad m = 0, 1, 2 \ldots$$

That is all odd terms vanish

For the even

$n=0$
$$a_2 = -\frac{1}{2 \cdot 3} a_0 = \frac{-1}{3!} a_0$$

$n=2$
$$a_4 = -\frac{1}{4 \cdot 5} a_2 = +\frac{1}{5!} a_0$$

$$\vdots$$

$$a_{2m} = \frac{(-1)^m}{(2m+1)!} a_0$$

Recall For $s = -1$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s} \qquad s = -1$$

$$= \sum_{m=0}^{\infty} a_{2m} x^{2m-1}$$

$$= x^{-1} a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}$$

$$= x^{-2} a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} x^{2m+1}$$

We immediately recognize the series!

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} x^{2m+1} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \equiv \sin x$$

Hence

$$\boxed{y(x) = a_0 \frac{\sin x}{x^2}} \qquad s = -1$$

is the solution for $s = -1$. There is a second linearly independent solution for $s = -2$ that is a HW problem.