REVIEW EVALS AN EVECTS

IF $\overleftrightarrow{M}$ IS A <u>SYMMETRIC</u> 3x3 MATRIX

THE EIGENVALUES CAN BE FOUND BY SOLVING

$$\det \left| \overleftrightarrow{M} - \lambda \overleftrightarrow{\mathbb{I}} \right| = 0$$

which gives rise to a <u>cubic</u> characteristic polynomical

$$A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$

WHICH CAN BE SOLVED NUMERICALLY OR

WITH CUBIC FORMULA FOR THREE ROOTS

λ_1 , λ_2 , λ_3 WHICH ARE EVALS.

$\Rightarrow$ $\exists$ A ROTATED COORDINATE SYSTEM

S.T. $\overleftrightarrow{M}' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ IS DIAGONAL

THE EIGEN VECTORS ARE $\hat{e}_1$ $\hat{e}_2$ $\hat{e}_3$

AND ARE FOUND BY SOLVING

$$\overleftrightarrow{M} \cdot \hat{e}_i = \lambda_i \hat{e}_i$$

WHICH GIVES 3 SETS OF 3 EQS & 3 UNKS.

BE SURE TO NORMALIZE SO $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

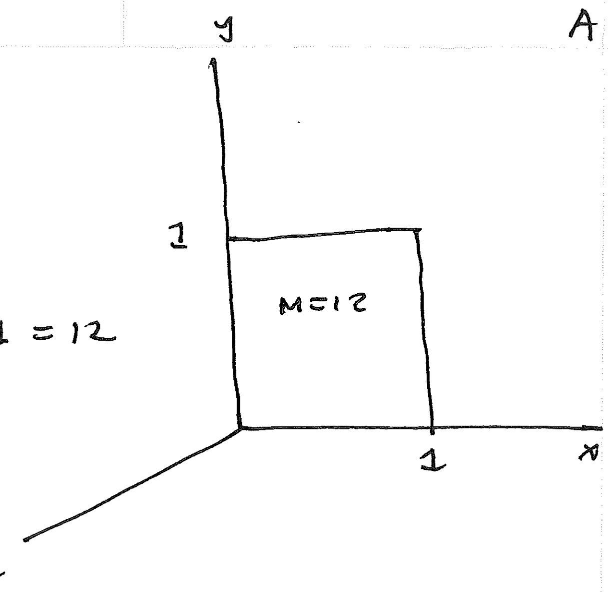
THE ROT MAT. $\overleftrightarrow{R}$ THAT TAKES $\overleftrightarrow{M} \rightarrow \overleftrightarrow{M}'$

IS CONSTRUCTED

$$\overleftrightarrow{R} = \begin{bmatrix} e_1^x & e_1^y & e_1^z \\ e_2^x & e_2^y & e_2^z \\ e_3^x & e_3^y & e_3^z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \ddots & \\ a_{31} & & \end{bmatrix}$$

AND FINALLY $\overleftrightarrow{M}' = \overleftrightarrow{R}^T \overleftrightarrow{M} \overleftrightarrow{R}$

2D Example



A thin metal plate of square size lies as shown in x-y plane WITH $M = 12$ and mass density $\rho(x;y) = 12/1\cdot 1 = 12$ uniform. Since $\rho(z) = 0$ this reduces to 2D.

$$I_{xx} = 12\int_0^1\int_0^1 dx\,dy\; y^2 = \frac{12}{3} = 4$$

$$I_{yy} = 12\int_0^1\int_0^1 dx\,dy\; x^2 = \frac{12}{3} = 4$$

$$I_{xy} = I_{yx} = -12\int_0^1\int_0^1 dx\,dy\; xy = -12\cdot\frac{1}{2}\cdot\frac{1}{2} = -3$$

$$\overleftrightarrow{I} = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

To find eigenvalues

$$\det|\overleftrightarrow{I} - \lambda\overleftrightarrow{\mathbb{1}}| = 0 = \det\begin{vmatrix} 4-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)^2 - 9 = 0 \quad \Rightarrow 16 - 8\lambda + \lambda^2 - 9 = 0$$

$$\Rightarrow \boxed{\lambda^2 - 8\lambda + 7 = 0}$$ Characteristic Polynomical

In general solve with <u>quadratic formula</u> but this one is rigged!

$$(4-\lambda)^2 = 9$$

$$\Rightarrow 4 - \lambda_\pm = \pm 3$$

$$\Rightarrow 4 \mp 3 = \lambda_\pm \quad \Rightarrow \boxed{\begin{matrix}\lambda_+ = 1 \\ \lambda_- = 7\end{matrix}}$$

There exists a rotated frame (x', y', z') in which $\overleftrightarrow{I}$ is diagonal! Let's Find it.

FIRST CONSTRUCT EVECTS

$$\overleftrightarrow{I} \cdot \hat{e}_+ = \lambda_+ \hat{e}_+$$

$$\overleftrightarrow{I} \cdot \hat{e}_- = \lambda_- \hat{e}_-$$

or
$$\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} e^x_+ \\ e^y_+ \end{bmatrix} = 1 \begin{bmatrix} e^x_+ \\ e^y_+ \end{bmatrix}$$

and
$$\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} e^x_- \\ e^y_- \end{bmatrix} = 7 \begin{bmatrix} e^x_- \\ e^y_- \end{bmatrix}$$

$\Rightarrow$
$$\begin{bmatrix} 4e^x_+ - 3e^y_+ = e^x_+ \\ -3e^x_+ + 4e^y_+ = e^y_+ \end{bmatrix}$$

and
$$\begin{bmatrix} 4e^x_- - 3e^y_- = 7e^x_- \\ -3e^x_- + 4e^y_- = 7e^y_- \end{bmatrix}$$

or
$$\begin{bmatrix} 3e^x_+ - 3e^y_+ = 0 \\ -3e^x_+ + 3e^y_+ = 0 \end{bmatrix} \; \text{\&} \; \begin{bmatrix} -3e^x_- - 3e^y_- = 0 \\ -3e^x_- - 3e^y_- = 0 \end{bmatrix}$$

That is two sets of homogeneous equations

BY INSPECTION WE SEE A SOLUTION

$$[\, e^x_+ = 1 \;\text{\&}\; e^y_+ = 1 \,] \;\text{\&}\; [\, e^x_- = -1 \;\text{\&}\; e^y_- = +1 \,]$$

Hence the eigenvectors (unnormalized) are

$$\hat{e}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{e}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

TO NORMALIZE NOTE $\sqrt{\hat{e}_+ \cdot \hat{e}_+} = \sqrt{2}$

and $\sqrt{\hat{e}_- \cdot \hat{e}_-} = \sqrt{2}$ SO JUST DIVIDE BY $\sqrt{2}$

NORMALIZED EVECTORS ARE

$$\hat{e}_+ = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \hat{e}_- = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\hat{e}_+ \cdot \hat{e}_+ = 1$ ✓
$\hat{e}_- \cdot \hat{e}_- = 1$ ✓
$\hat{e}_+ \cdot \hat{e}_- = 0$ ✓

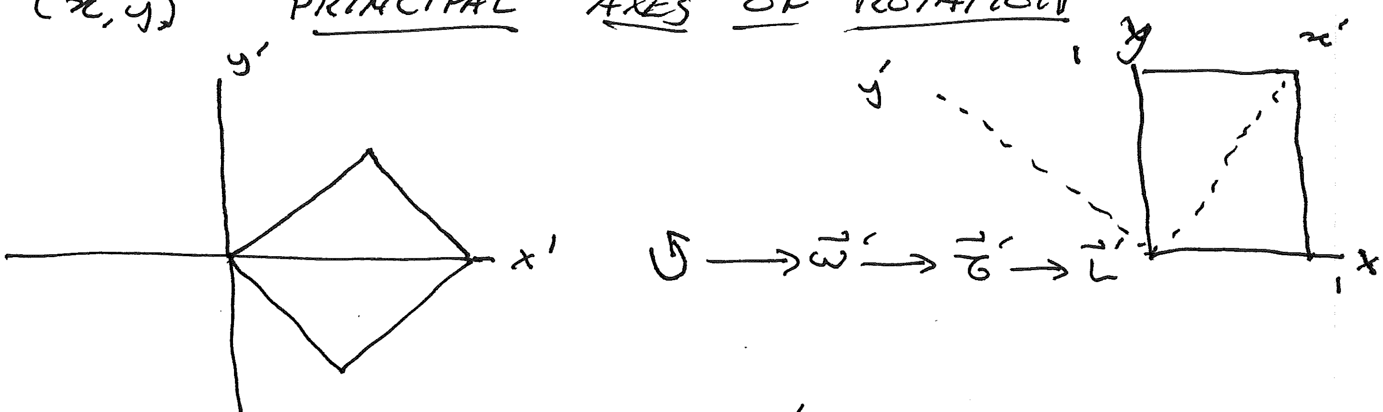
ORTHO NORMAL SET

WE CONSTRUCT THE ROT. MATRIX

$$\overleftrightarrow{R} = \left[\left(\hat{e}_+ \right) \left(\hat{e}_- \right) \right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

THAT IS $\overleftrightarrow{R} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

Hence $\vec{e}_\pm$ point in direction of new (x', y') PRINCIPAL AXES OF ROTATION



WE COULD HAVE GUESSED THIS! IT WILL SPIN LIKE A TOP ON ONE CORNER! FINALLY

$$\overleftrightarrow{I}' = \overleftrightarrow{R}^T \overleftrightarrow{I} \overleftrightarrow{R}$$

CHECK

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

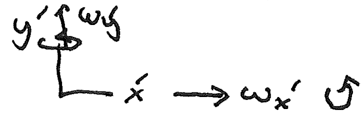
$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4-3 & -4-3 \\ -3+4 & 3+4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 1 & 7 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$ ✓

$I'_{xx} = 1$ AND $I'_{yy} = 7$ ARE PRINCEPAL MOMENTS OF INERTIA

IN NEW FRAME EQ's SIMPLIFY

$$\vec{L}' = \overleftrightarrow{I}' \cdot \overleftrightarrow{\omega}' = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} \omega_x' \\ \omega_y' \end{bmatrix} = \begin{bmatrix} \omega_x' \\ 7\omega_y' \end{bmatrix}$$



Hence

$$\vec{L}' = \omega_{x}' \, \hat{x}' + 7\omega_y \, \hat{j}'$$

That is rotation can now be decomposed into (spin about x') + (spin about y') with no "tumbling".