A Schrödinger Equation for Solving the Bender-Brody-Müller Conjecture

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Abstract. The Hamiltonian of a quantum mechanical system has an affiliated spectrum. If this spectrum is the sequence of prime numbers, a connection between quantum mechanics and the nontrivial zeros of the Riemann zeta function can be made. In this case, the Riemann zeta function is analogous to chaotic quantum systems, as the harmonic oscillator is for integrable quantum systems. Such quantum Riemann zeta function analogies have led to the Bender-Brody-Müller (BBM) conjecture, which involves a non-Hermitian Hamiltonian that maps to the zeros of the Riemann zeta function. If the BBM Hamiltonian can be shown to be Hermitian, then the Riemann Hypothesis follows. As such, herein we perform a symmetrization procedure of the BBM Hamiltonian to obtain a unique Hermitian Hamiltonian that maps to the zeros of the analytic continuation of the Riemann zeta function, and discuss the eigenvalues of the results. Moreover, a second quantization of the resulting Schrödinger equation is performed, and a convergent solution for the nontrivial zeros of the analytic continuation of the Riemann zeta function is obtained. Finally, the Hilbert-Pólya conjecture is discussed, and it is heuristically shown that the real part of every nontrivial zero of the Riemann zeta function converges at $\sigma = 1/2$.

INTRODUCTION

The unification of number theory with quantum mechanics has been the subject of many research investigations [1, 2, 3, 4, 5]. It has been proven that an infinitude of prime numbers exist [6]. In Ref. [7], it was shown that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeros of the Riemann zeta function. If the Riemann hypothesis is correct [8], the zeros of the Riemann zeta function can be considered as the spectrum of an operator $\hat{R} = \hat{I}/2 + i\hat{H}$, where $\hat{H}$ is a self-adjoint Hamiltonian operator [5, 9], and $\hat{I}$ is identity. Hilbert proposed the Riemann Hypothesis as the eighth problem on a list of significant mathematics problems [10]. Although the BBM Hamiltonian is pseudo-Hermitian [11], it is consistent with the Berry-Keating conjecture [12, 13, 14], which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$. Berry, Keating and Connes proposed the classical Hamiltonian in an effort to map the Riemann zeros to the Hamiltonian spectrum. More recently, the classical Berry-Keating Hamiltonians were quantized, and were found to contain a smooth approximation of the Riemann zeros [15, 16]. This reformulation was found to be physically equivalent to the Dirac equation in Rindler spacetime [17]. Herein, the eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the nontrivial zeroes of the analytical continuation of the Riemann zeta function

$$\zeta(s) = \frac{1}{1-2^{1-s}} \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{1}$$

where the complex number $s = \sigma + it$, and $\Re(s) > 0$. The idea that the imaginary parts of the zeros of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [18]. Hilbert and Pólya asserted that the nontrivial zeros of Eq. (1) can be considered as the spectrum of a self-adjoint operator in a suitable Hilbert space. The Hilbert-Pólya conjecture has also found applications in quantum field theories [19]. The Riemann Hypothesis states

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that the zeros of Eq. (1) on $0 \leq \sigma < 1$ have real part equal to $1/2$ [8, 20]. In Ref. [21], Hardy proved that infinitely many zeros are located at $\sigma = 1/2$. According to the Prime Number Theorem [22, 23], no zeros of Eq. (1) can exist at $\sigma = 1$. In this paper we present a Schrödinger equation that maps to the nontrivial zeros of the Riemann zeta function, and evaluate the convergence of the expression by imposing a normalization constraint on the density. A self-adjoint Hamiltonian is derived from the BBM Hamiltonian, and a second quantization of the resulting Schrödinger equation is performed to obtain the equations of motion. Finally, the Hilbert-Pólya conjecture is discussed, and we study the nontrivial zeros of the Riemann zeta function by taking the expectation values of the resulting Schrödinger equation to show that the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function converges at $\sigma = 1/2$.

RIEMANN ZETA SCHRÖDINGER EQUATION

We consider the eigenvalues of the Hamiltonian

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}), \tag{2}$$

where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$. In Ref. [7], it is conjectured that if the Riemann hypothesis is correct, the eigenvalues of Eq. (2) are non-degenerate. Next, we let $\Psi_s(x)$ be an eigenfunction of Eq. (2) with an eigenvalue $t = i(2s - 1)$, such that

$$\hat{H}\,|\Psi_s(x)\rangle = t\,|\Psi_s(x)\rangle\,, \tag{3}$$

and $x \in \mathbb{R}^+$, $s \in \mathbb{C}$. Solutions to Eq. (3) are given by the analytic continuation of the Hurwitz zeta function

$$\begin{aligned} |\Psi_s(x)\rangle &= -\zeta(s, x+1) \\ &= -\Gamma(1-s)\frac{1}{2\pi i}\oint_C \frac{z^{s-1}e^{(x+1)z}}{1-e^z}dz, \end{aligned} \tag{4}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2s-1)$, $s \in \mathbb{C}$, $\Re(s) \leq 1$, the contour C is a loop around the negative real axis, and Γ is the Euler gamma function for $\Re(s) > 0$

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx. \tag{5}$$

As $-\,|\Psi_s(x = 1)\rangle$ is $1 - \zeta(s^*)$, this implies that s belongs to the discrete set of zeros of the Riemann zeta function when $s^* = \sigma - it$. From inserting Eq. (3) into Eq. (2), we have the relation

$$\frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\,|\Psi_s(x)\rangle = t\,|\Psi_s(x)\rangle\,. \tag{6}$$

Given that Eq. (2) is not Hermitian, it is useful to symmetrize the system. This can be accomplished by letting

$$\begin{aligned} |\phi_s(x)\rangle &= [1 - \exp(-\partial_x)]\,|\Psi_s(x)\rangle\,, \\ &= \hat{\Delta}\,|\Psi_s(x)\rangle \\ &= |\Psi_s(x)\rangle - |\Psi_s(x-1)\rangle\,, \end{aligned} \tag{7}$$

and defining a shift operator

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x). \tag{8}$$

For $s > 0$ the only singularity of $\zeta(s, x)$ in the range of $0 \leq x \leq 1$ is located at $x = 0$, behaving as x^{-s}. More specifically,

$$\zeta(s, x+1) = \zeta(s, x) - \frac{1}{x^s}, \tag{9}$$

with $\zeta(s,x)$ finite for $x \geq 1$ [25]. As such, it can be seen from Eq. (7) that the eigenfunction

$$|\phi_s(x)\rangle = \frac{1}{x^s}. \tag{10}$$

Upon inserting Eq. (7) into Eq. (6) we obtain

$$-i[x\partial_x + \partial_x x]\,|\phi_s(x)\rangle = t\,|\phi_s(x)\rangle\,. \tag{11}$$

Let $\mathscr{H}$ be a Hilbert space, and from Eq. (11) we have the Hamiltonian operator

$$\begin{aligned}\hat{H} &= -i\hbar\Big[x\partial_x + \partial_x x\Big]\\ &= -i\hbar\Big[2x\partial_x + 1\Big],\end{aligned} \tag{12}$$

for $x \in \mathbb{R}$ acting in $\mathscr{H}$, such that

$$\langle \hat{H}f, g\rangle = \langle f, \hat{H}g\rangle\ \ \forall\ f, g \in \mathscr{D}(\hat{H}). \tag{13}$$

Restricting $x \in \mathbb{R}^+$, Eq. (12) is then written

$$\hat{H} = -2i\hbar\,\sqrt{x}\partial_x\,\sqrt{x}, \tag{14}$$

where $s \in \mathbb{C}$, and $x \in \mathbb{R}^+$. For the Hamiltonian operator as given by Eq. (14), the Hilbert space is $\mathscr{H} = L^2(0,\infty)$ [26, 27, 28]. We then impose on Eq. (14) the following minimal requirements, such that its domain is not too artificially restricted.

- i $\hat{H}$ is a symmetric (Hermitian) linear operator;
- ii $\hat{H}$ can be applied on all functions of the form

$$g(x,s) = P(x,s)\exp\Big(-\frac{x^2}{2}\Big), \tag{15}$$

where P is a polynomial of x and s. Here, it should be pointed out that $\hat{H} = \hat{T} + \hat{V}$, and from Eq. (12), it can be seen that $\hat{T} = -2i\hbar x\partial_x$, $\hat{V} = -i\hbar$. From (ii), $\hat{V}g(x,s)$ must belong to the Hilbert space $\mathscr{H} = L^2$ defined over the space x. This is guaranteed as

$$|-i\hbar| \leq C, \tag{16}$$

where C is constant. The domain $\mathscr{D}_{\hat{V}}$ of the potential energy $\hat{V}$ consists of all $\phi \in \mathscr{H}$ for which $\hat{V}\phi \in \mathscr{H}$. As such, $\hat{V}$ is self-adjoint. It is not necessary to specify the domain of Eq. (14), as it is only necessary to admit that Eq. (14) is defined on a certain $\mathscr{D}_{\hat{H}}$ such that (i) and (ii) are satisfied. If we denote by $\mathscr{D}_1$ the set of all functions in Eq. (15), then (ii) implies that $\mathscr{D}_{\hat{H}} \supseteq \mathscr{D}_1$. By letting $\hat{H}_1$ be the contraction of $\hat{H}$ with domain $\mathscr{D}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, and letting $\tilde{H}_1$ be the closure of $\hat{H}$, it can be seen that $\tilde{H}_1$ is self-adjoint. Since $\hat{H}$ is symmetric and $\hat{H} \supseteq \hat{H}_1$, i.e., $\hat{H}$ is an extension of $\hat{H}_1$, it follows that $\tilde{H} = \tilde{H}_1$ and $\hat{H}$ is essentially self-adjoint, where $\tilde{H}$ is the unique self-adjoint extension [29]. Other than eigenfunctions $\phi_s(x)$ in configuration space as seen in Eq. (10), it is useful to represent eigenfunctions in momentum space $\Phi_s(p)$. The transformation between configuration space eigenfunctions and momentum space eigenfunctions can be obtained via Plancherel transforms [30], where the one-to-one correspondence $\phi_s(x) \rightleftharpoons \Phi_s(p)$ is linear and isometric, i.e.,

$$\|\,\phi\,\|^2 \equiv \int_{-\infty}^{-1} |\,\phi_s(x)\,|^2\,dx + \int_1^{\infty} |\,\phi_s(x)\,|^2\,dx = \int_{-\infty}^{-1} |\,\Phi_s(p)\,|^2\,dp + \int_1^{\infty} |\,\Phi_s(p)\,|^2\,dp \equiv \|\,\Phi\,\|^2, \tag{17}$$

from which

$$\begin{aligned}\Phi_s(p) &= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\infty}\phi_s(x)\exp(-ipx)dx\\ &= \frac{e^{-\frac{1}{2}i\pi s}(\mathrm{sgn}(p)+1)\sin(\pi s)\Gamma(1-s)\,|p|^{s-1}}{2\pi^{3/2}},\ 0 < \Re(s) < 1,\end{aligned} \tag{18}$$

$$\phi_s(x) = \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\infty}\Phi_s(p)\exp(ipx)dp = \frac{1}{x^s}, \tag{19}$$

and

$$\| \phi \|^2 = \| \Phi \|^2 = \frac{(-1)^{-2\sigma}}{2\sigma - 1} + \frac{1}{2\sigma - 1}, \tag{20}$$

where $\sigma > 1/2$. Interestingly, it can be seen that

$$\lim_{\sigma \to 1/2} \left[\frac{(-1)^{-2\sigma}}{2\sigma - 1} + \frac{1}{2\sigma - 1} \right] = i\pi. \tag{21}$$

Eqs. (18) and (19) are two vector representations of the same Hilbert space $\mathscr{H}$. From Eq. (12), it can be seen that

$$\hat{T} = -2i\hbar x \partial_x, \tag{22}$$

such that we define a multiplicative operator $\hat{T}_0$ in momentum space $(\hat{T}_0 \Phi_s)(p) = \hat{T}_0(p)\Phi_s(p)$, where

$$\hat{T}_0(p) = 2\hat{x}\hat{p}. \tag{23}$$

Here, it should be pointed out that as $\hat{x} = i\hbar d/dp$, Eq. (23) reduces to

$$\hat{T}_0(p) = 2i\hbar, \tag{24}$$

and Eq. (12) is then rewritten in momentum space as $\hat{H}(p) = i\hbar$. The domain $\mathscr{D}_0$ of $\hat{T}_0$ is defined as the set of all functions $\Phi_s(p) \in \mathscr{H}$ such that $\hat{T}_0(p)\Phi_s(p) \in \mathscr{H}$. As such, $\hat{T}_0$ is definitively self-adjoint. From Eq. (15) we have defined the set $\mathscr{D}_1$ of functions in configuration space. From the Plancherel transform [30] of Eq. (15), we obtain the set $\mathscr{D}_1$ of functions in momentum space having the form

$$G(p, s) = P(p, s) \exp\left(-\frac{p^2}{2} \right), \tag{25}$$

where P is a polynomial of p and s. Eqs. (18) and (19) are true if $\phi_s(x) \in \mathscr{D}_1$ or $\Phi_s(p) \in \mathscr{D}_1$ and since $\Phi_s(p) \in \mathscr{D}_1 \to 0$ as $p \to \infty$ then $\mathscr{D}_1 \subseteq \mathscr{D}_0$. Moreover, for $\phi \in \mathscr{D}_1$, $\hat{T}_0$ coincides with Eq. (22) [29].

Definition 1 *The Riemann zeta Schrödinger equation is*

$$-\hbar \partial_s |\Psi_s(x)\rangle = i \left[\hat{\Delta}^{-1} \hat{x}\hat{p}\hat{\Delta} + \hat{\Delta}^{-1} \hat{p}\hat{x}\hat{\Delta} \right] |\Psi_s(x)\rangle, \tag{26}$$

where $\hat{\Delta}$ is given by Eq. (8), $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+ \geq 1$ owing to the difference operator $\hat{\Delta}|\Psi_s(x)\rangle$, and $s \in \mathbb{C}$.

Upon inserting Eq. (7) into Eq. (26) for $x \in \mathbb{R}^+$, we obtain the symmetrized Riemann zeta Schrödinger equation, i.e.,

$$\begin{aligned} \partial_s |\phi_s(x)\rangle &= 1/2(\partial_\sigma - i\partial_t) |\phi_s(x)\rangle \\ &= -\frac{2}{\hbar} \sqrt{x} \partial_x \sqrt{x} |\phi_s(x)\rangle. \end{aligned} \tag{27}$$

Theorem 1 The eigenvalues t of $\hat{H} = -2i\hbar \sqrt{x}\partial_x \sqrt{x}$ are real for $\hbar = 1$ and $|\phi_s(x)\rangle = x^{-s}$.

Proof 1 *Let $|\phi_s(x)\rangle$ be an eigenfunction of $\hat{H}$ with eigenvalue t, i.e.,*

$$\hat{H} |\phi_s(x)\rangle = t |\phi_s(x)\rangle. \tag{28}$$

In order to find the expectation value for $\hat{H}$ we take the complex conjugate of Eq. (28), multiply by the eigenfunction, and then integrate over $x \in [1, \infty)$, $[-1, -\infty)$ to obtain

$$\begin{aligned} 2i \int_{-\infty}^{-1} \left(\sqrt{x}\partial_x \sqrt{x}\phi_s(x) \right)^* \phi_s(x) dx + 2i \int_1^{\infty} \left(\sqrt{x}\partial_x \sqrt{x}\phi_s(x) \right)^* \phi_s(x) dx &= t^* \int_{-\infty}^{-1} \phi_s^*(x)\phi_s(x) dx \\ &+ t^* \int_1^{\infty} \phi_s^*(x)\phi_s(x) dx \\ &= t^* \| \phi \|^2. \end{aligned} \tag{29}$$

Integrating by parts on the LHS then gives

$$-2i\left(\parallel \phi \parallel^2 + \int_{-\infty}^{-1} \phi_s^*(x) x \frac{d}{dx}\phi_s(x)dx + \int_1^{\infty} \phi_s^*(x) x \frac{d}{dx}\phi_s(x)dx\right) = t^* \parallel \phi \parallel^2 . \tag{30}$$

Carrying out the integration on the LHS we obtain

$$\frac{2i(-1)^{-2\sigma}\left((-1)^{2\sigma}+1\right)(\sigma+it)}{2\sigma-1} = (t^*+2i) \parallel \phi \parallel^2 . \tag{31}$$

Upon inserting Eq. (20) it can be seen that

$$2i(-1)^{-2\sigma}\left((-1)^{2\sigma}+1\right)(\sigma+it) = (t^*+2i)\Big((-1)^{-2\sigma}+1\Big), \tag{32}$$

Hence,

$$\sigma = \pm\frac{1}{2} \ \forall \ t, t^*. \tag{33}$$

Second Quantization

Theorem 2 If all of the nontrivial zeros of $\zeta(s) = (1-2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ occur at $s = 1/2 + it_n$, where $t_n = 2n-1$, then the numbers t_n correspond to the eigenvalues (spectrum) of $\hat{H} = -2\hbar\sqrt{x}\partial_x\sqrt{x}$, and $n \in \mathbb{Z} \subseteq \mathbb{R}$.

Proof 2 *In order to perform a second quantization [31], we can express the complex order parameter as a linear combination of basis states*

$$|\phi_s(x)\rangle = \sum_n b_n(s)\,|\phi_n(x)\rangle, \tag{34}$$

where $n \in \mathbb{Z} \subseteq \mathbb{R}$. From Eq. (27) we find

$$\begin{aligned} -\hbar\frac{d}{ds}b_n(s) &= t_n b_n(s) \\ &= (1-2n)b_n(s). \end{aligned} \tag{35}$$

We now find a Hamiltonian that yields Eq. (35) as the equation of motion. Hence, we take

$$\begin{aligned} \langle\phi_s(x)|\,\hat{H}\,|\phi_s(x)\rangle &= 2\int_1^{\infty}\phi_s^*(x)\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\,dx \\ &= 2\,\langle\phi_s(x)|\,\widehat{\sqrt{x}\partial_x\sqrt{x}}\,|\phi_s(x)\rangle, \end{aligned} \tag{36}$$

as the expectation value. Upon substituting Eq. (34) into Eq. (36), we obtain the harmonic oscillator

$$\begin{aligned} \langle\phi_m(x)|\,\hat{H}\,|\phi_n(x)\rangle &= \sum_m\sum_n(1-2m)b_m^*(s)b_n(s)\int_1^{\infty}\frac{1}{x^{m+n}}dx \\ &= \sum_m\sum_n(1-2m)\hat{b}_m^{\dagger}(s)\hat{b}_n(s)\frac{1}{m+n-1}P_n, \end{aligned} \tag{37}$$

for $m+n > 1$, $P_n = 1, 2, 3, \ldots, \infty$. Taking $b_n(s)$ as an operator, and $b_n^*(s)$ as the adjoint, we obtain the usual properties:

$$\begin{aligned} [\hat{b}_n(s), \hat{b}_m(s)] &= [\hat{b}_n^{\dagger}(s), \hat{b}_m^{\dagger}(s)] = 0, \\ [\hat{b}_n(s), \hat{b}_m^{\dagger}(s)] &= \delta_{nm}. \end{aligned} \tag{38}$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned}
-\hbar\frac{d}{ds}\hat{b}_n(s) &= [\hat{b}_n(s), \hat{H}]_- \\
&= \sum_m E_m\Big(\hat{b}_n(s)\hat{b}_m^\dagger(s)\hat{b}_m(s) - \hat{b}_m^\dagger(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= \sum_m E_m\Big(\delta_{nm}\hat{b}_m(s) - \hat{b}_m^\dagger(s)\hat{b}_n(s)\hat{b}_m(s) - \hat{b}_m^\dagger(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= \sum_m E_m\Big(\delta_{nm}\hat{b}_m(s) + \hat{b}_m^\dagger(s)\hat{b}_m(s)\hat{b}_n(s) - \hat{b}_m^\dagger(s)\hat{b}_m(s)\hat{b}_n(s)\Big) \\
&= t_n\hat{b}_n(s) \\
&= (1-2n)\hat{b}_n(s). \qquad (39)
\end{aligned}$$

The eigenvalues of $\hat{H}$ are then

$$\begin{aligned}
\langle\phi_m(x)|\,\hat{H}\,|\phi_n(x)\rangle &= \sum_n t_n P_n \\
&= \sum_n (1-2n)P_n. \qquad (40)
\end{aligned}$$

Remark 1 *Theorem 2 implies the Riemann hypothesis, as the spectrum of a Hermitian operator consists of real numbers as seen in Theorem 1, and $n \in \mathbb{Z} \subseteq \mathbb{R}$.*

Convergence

Theorem 3 The eigenfunction $\phi_s(x) = x^{-s}$ of $\hat{H} = -2i\hbar\sqrt{x}\partial_x\sqrt{x}$ normalizes at $x = 1$.

Proof 3 *In order to study convergent solutions to the Riemann zeta Schrödinger Eq. (27), it can be seen that upon inserting Eq. (7) into Eq. (27), we obtain*

$$\begin{aligned}
s &= \sigma + it \\
&= \frac{1}{2} - \frac{\log(x)}{2}. \qquad (41)
\end{aligned}$$

Hence, at $x = 1$

$$\sigma = \frac{1}{2} - it, \qquad (42)$$

and Eq. (37) becomes

$$\langle\phi_m|\,\hat{H}\,|\phi_n\rangle = \sum_m\sum_n (1-2m)\hat{b}_m^\dagger(s)\hat{b}_n(s)P_n. \qquad (43)$$

This condition is required such that the density is normalized, i.e.,

$$\begin{aligned}
\langle\phi_s(x)|\phi_s(x)\rangle &= \Big(\Gamma(1-s)\frac{1}{2\pi i}\oint_C z^{s-1}dz\Big)^*\Big(\Gamma(1-s)\frac{1}{2\pi i}\oint_C z^{s-1}dz\Big) \\
&= 1 \\
&= \sum_m\sum_n b_n(s)b_m^*(s)\,\langle\phi_m|\phi_n\rangle \\
&= \sum_n |b_n(s)|^2. \qquad (44)
\end{aligned}$$

The matrix elements of Eq. (37) in the eigenbasis $|\phi_n(x)\rangle$ are:

$$\begin{aligned}\hat{H}_{mn} &= -2\langle\phi_m(x)|\sqrt{x}\partial_x\sqrt{x}\phi_n(x)\rangle \\ &= -2\langle\sqrt{x}\partial_x\sqrt{x}\phi_m(x)|\phi_n(x)\rangle \\ &= -2\langle\phi_m(x)|\sqrt{x}\partial_x\sqrt{x}|\phi_n(x)\rangle\,. \end{aligned} \tag{45}$$

The expectation value of $\hat{H}$ at s for the system in state $|\phi_s(x)\rangle$ is:

$$\begin{aligned}\langle\hat{H}\rangle &= \langle\phi_s(x)|\hat{H}\phi_s(x)\rangle \\ &= \sum_m\sum_n b_m^\dagger(s)b_n(s)\hat{H}_{mn}. \end{aligned} \tag{46}$$

Here, it should be pointed out that $\langle\hat{H}\rangle$ is an expansion quadratic in the b_n coefficients. From Eq. (37) it can be seen that at the density normalization constraint $x = 1$, the matrix elements in the eigenbasis are

$$\hat{H}_{mn} = (1-2m)\hat{b}_m^\dagger\hat{b}_n, \tag{47}$$

such that we obtain the eigenequation

$$(1-2m)\hat{b}_m^\dagger\hat{b}_n\mathbb{1} = (1-2n)\mathbb{1}. \tag{48}$$

The trace of $\hat{H}_{mn}$ is also the sum of all eigenvalues t_n

$$\mathrm{Tr}(\hat{H}_{mn}) = \sum_n(1-2n), \tag{49}$$

and the determinant of $\hat{H}_{mn}$ is the product of all eigenvalues t_n

$$\det(\hat{H}_{mn}) = \prod_n(1-2n). \tag{50}$$

More generally,

$$\mathrm{Tr}(\hat{H}_{mn}^k) = \sum_n(1-2n)^k. \tag{51}$$

The characteristic equation of $\hat{H}_{mn}$ is

$$|\hat{H}_{mn} - t_n\mathbb{1}_{mn}| = (1-2n) - t_n \tag{52}$$

which has roots $t_n = 1 - 2n$.

Theorem 4 Imaginary solutions of $\hbar\partial_s|\phi_s(x)\rangle = -2\sqrt{x}\partial_x\sqrt{x}|\phi_s(x)\rangle$ at $x = 1$ map to the nontrivial zeros of $\zeta(s) = (1-2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$.

Proof 4 *At $x = 1$, the normalization constraint Eq. (44) is satisfied, $\sigma = \frac{1}{2} - it$, and Eq. (4) can be written*

$$\begin{aligned}\Psi_s(x=1) &= -\zeta(\sigma, 2) \\ &= -\Gamma(1-s)\frac{1}{2\pi i}\oint_C\frac{z^{s-1}e^{2z}}{1-e^z}dz \\ &= 1 - \zeta(\sigma = \frac{1}{2} - it). \end{aligned} \tag{53}$$

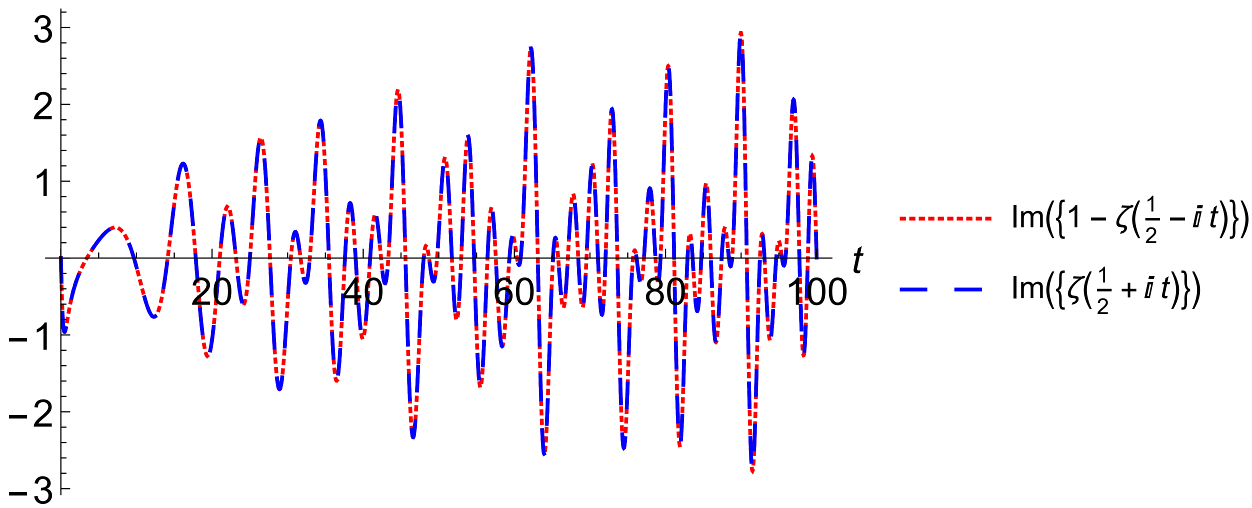

FIGURE 1. Plot of the imaginary components of Eq. (1). Results are compared with Eq. (56) (color online).

where the contour C is about $\mathbb{R}^-$*. From the analytic continuation relations of Eq. (1)*

$$\begin{aligned}\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} &= \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(-i\cdot t\ln(n)\Big)}{\sqrt{n}}\\ &= \frac{1}{1-2^{1/2-it}}\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{\sqrt{n}} - i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{\sqrt{n}}\Big],\end{aligned} \tag{54}$$

$$\begin{aligned}1-\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^* &= 1-\frac{1}{1-2^{1-s^*}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\Big(i\cdot t\ln(n)\Big)}{\sqrt{n}}\\ &= \Big(1-\frac{1}{1-2^{1/2+it}}\Big)\Big[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos\Big(t\cdot\ln(n)\Big)}{\sqrt{n}} + i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin\Big(t\cdot\ln(n)\Big)}{\sqrt{n}}\Big],\end{aligned} \tag{55}$$

and from Fig. 1 it can be seen that

$$\Im\Big[\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big] = \Im\Big[1-\Big(\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\Big)^*\Big]. \tag{56}$$

Finally, when $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 0$*, we obtain*

$$\Im(1) = 0. \tag{57}$$

However, since $t_n = i(\sigma_n - 1/2)$ at the normalization constraint $x = 1$, we can rewrite Eq. (40) as

$$\langle\phi_s(x)|\,\hat{H}\,|\phi_s(x)\rangle = i\sum_n\Big(\sigma_n - \frac{1}{2}\Big)P_n. \tag{58}$$

From Eq. (39) it can be seen that

$$\begin{aligned}-\hbar\frac{d}{ds}\hat{b}_n &= i\Big(\sigma_n - \frac{1}{2}\Big)\hat{b}_n,\\ -\hbar\frac{d}{ds}\hat{b}_m^\dagger &= -i\Big(\sigma_m - \frac{1}{2}\Big)\hat{b}_m^\dagger.\end{aligned} \tag{59}$$

Finally, upon setting $t_n = i(\sigma_n - 1/2) = (1 - 2n)$ and $t_m = -i(\sigma_m - 1/2) = (1 - 2m)$, we obtain

$$\begin{aligned} \hbar\frac{d}{ds}\hat{b}_n &= (2n-1)\hat{b}_n, \\ \hbar\frac{d}{ds}\hat{b}_m^\dagger &= (2m-1)\hat{b}_m^\dagger. \end{aligned} \tag{60}$$

Eq. (39) is in agreement with Eq. (60), and can be solved using the Wirtinger derivatives. For purposes of clarity, we can also set Eq. (37) equal to Eq. (58) to obtain

$$i\sum_n\left(\sigma_n - \frac{1}{2}\right)P_n = \sum_m\sum_n(1-2m)\hat{b}_m^\dagger(s)\hat{b}_n(s)\frac{1}{m+n-1}P_n. \tag{61}$$

This gives the relation

$$\begin{aligned} \sum_n \sigma_n P_n &= \sum_n \frac{1}{2}P_n - i\sum_{m,n}(1-2m)\hat{b}_m^\dagger(s)\hat{b}_n(s)\frac{1}{m+n-1}P_n \\ &= \sum_n \frac{1}{2}P_n - i\sum_m\sum_n\frac{1-2m}{m+n-1}P_n\delta_{nm}. \end{aligned} \tag{62}$$

Density Operator

Consider the operator $|\phi_s(x)\rangle\langle\phi_s(x)|$ with matrix elements

$$\langle\phi_m(x)|\phi_s(x)\rangle\langle\phi_s(x)|\phi_n(x)\rangle = \hat{b}_m(s)\hat{b}_n^\dagger(s). \tag{63}$$

These matrix elements are required for the calculation of $\langle\hat{H}\rangle$. We take the Hermitian density operator as

$$\hat{\rho}(s) \equiv |\phi_s(x)\rangle\langle\phi_s(x)|, \tag{64}$$

with matrix elements

$$\begin{aligned} \hat{\rho}_{mn}(s) &= \langle\phi_m(x)|\hat{\rho}(s)\phi_n(x)\rangle \\ &= \hat{b}_m(s)\hat{b}_n^\dagger(s). \end{aligned} \tag{65}$$

At $x = 1$

$$\begin{aligned} \sum_n \hat{\rho}_{nn}(s) &= \mathrm{Tr}[\hat{\rho}(s)] \\ &= \sum_n |\hat{b}_n(s)|^2 \\ &= \mathbb{1}. \end{aligned} \tag{66}$$

The expectation value of $\hat{H}$ can be expressed via the density operator, viz.,

$$\begin{aligned} \langle\hat{H}\rangle(s) &= \sum_m\sum_n \hat{b}_m(s)\hat{b}_n^\dagger(s)(1-2m) \\ &= \sum_m\sum_n \hat{\rho}_{nm}(s)(1-2m) \\ &= \sum_n [\hat{\rho}(s)(1-2n)]_{nn} \\ &= \mathrm{Tr}[\hat{\rho}(s)(1-2n)] \\ &= 1-2n. \end{aligned} \tag{67}$$

NONTRIVIAL RIEMANN ZEROS

Theorem 5 The real part of all of the *nontrivial* zeros of $\zeta(s) = (1-2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ are located at $\sigma = 1/2$.

Proof 5 *To find σ, we integrate Eq. (11) over $x \in \mathbb{R}^+ \geq 1$ owing to the difference operator $\hat{\Delta}\Psi_s(x)$, to obtain the eigenvalue*

$$t^* = \frac{i}{\langle\phi_s(x)|\phi_s(x)\rangle}\int_1^\infty \left(x\partial_x\phi_s(x)\right)^*\phi_s(x)dx + \frac{i}{\langle\phi_s(x)|\phi_s(x)\rangle}\int_1^\infty \left(\partial_x x\phi_s(x)\right)^*\phi_s(x)dx. \tag{68}$$

From compact support, it can be seen that $\phi_s(x)$ vanishes as $x \to \infty$. Next we integrate the first term on the RHS of Eq. (68) by parts to obtain the inner product

$$\langle\phi_s(x)|\phi_s(x)\rangle = -\int_1^\infty x\phi_s(x)\partial_x\phi_s^*(x)dx - \int_1^\infty \phi_s^*(x)x\frac{d}{dx}\phi_s(x)dx, \tag{69}$$

and integrate the second term on the RHS of Eq. (68) by parts to obtain the inner product

$$\langle\phi_s(x)|\phi_s(x)\rangle = -\int_1^\infty x\phi_s^*(x)\partial_x\phi_s(x)dx - \int_1^\infty \phi_s(x)x\frac{d}{dx}\phi_s^*(x)dx. \tag{70}$$

Upon substituting Eqs. (69) and (70) into Eq. (68), we obtain

$$(it^* - 2)\langle\phi_s(x)|\phi_s(x)\rangle = \langle\phi_s(x)|\,\widehat{x\tfrac{d}{dx}}\,|\phi_s(x)\rangle + |\phi_s(x)\rangle\,\widehat{x\tfrac{d}{dx}}\,\langle\phi_s(x)|, \tag{71}$$

where

$$\langle\phi_s(x)|\phi_s(x)\rangle = \int_1^\infty \phi_s^*(x)\phi_s(x)dx, \tag{72}$$

and

$$|\phi_s(x)\rangle\,\widehat{x\tfrac{d}{dx}}\,|\phi_s(x)\rangle = \int_1^\infty \phi_s(x)x\frac{d}{dx}\phi_s(x)dx. \tag{73}$$

Next, we rewrite $|\phi_s(x)\rangle$, such that

$$|\phi_s(x)\rangle = |\phi_\sigma(x)\rangle + i\,|\phi_t(x)\rangle, \tag{74}$$

and substitute Eq. (74) into Eq. (71) to obtain

$$t^* = -\frac{2i}{\langle\phi_s(x)|\phi_s(x)\rangle}|\phi_\sigma(x)\rangle\,\widehat{x\tfrac{d}{dx}}\,|\phi_\sigma(x)\rangle - \frac{2i}{\langle\phi_s(x)|\phi_s(x)\rangle}|\phi_t(x)\rangle\,\widehat{x\tfrac{d}{dx}}\,|\phi_t(x)\rangle - 2i. \tag{75}$$

Taking $t = i(\sigma - 1/2)$ and $|\phi_\sigma(x)\rangle = |\phi_t(x)\rangle$ in Eq. (75), it can be seen that

$$\sigma = \frac{4}{\langle\phi_s(x)|\phi_s(x)\rangle}\int_1^\infty \phi_s(x)x\frac{d}{dx}\phi_s(x)dx + \frac{5}{2}. \tag{76}$$

For $x \in \mathbb{R}^+ \geq 1$, Eq. (11) can then be written

$$\begin{aligned} -2i\sqrt{x}\partial_x\sqrt{x}\phi_s(x) &= t\phi_s(x) \\ &= i(\sigma - 1/2)\phi_s(x). \end{aligned} \tag{77}$$

To find σ, we integrate over $x \in \mathbb{R}^+ \geq 1$ owing to the difference operator $\hat{\Delta}\Psi_s(x)$, to obtain the eigenvalue

$$t^* = \frac{2i}{\langle\phi_s(x)|\phi_s(x)\rangle}\int_1^\infty \left(\sqrt{x}\partial_x\sqrt{x}\phi_s(x)\right)^*\phi_s(x)dx. \tag{78}$$

Since $\sqrt{x}\hat{p}\sqrt{x}$ and $\hat{x}\hat{p}$ commute, we obtain

$$t^* = \frac{2i}{\langle \phi_s(x)|\phi_s(x)\rangle} \int_1^\infty \Big(x\partial_x \phi_s(x)\Big)^* \phi_s(x) dx. \tag{79}$$

By again using Eq. (69) in Eq. (79), we obtain the inner product

$$\langle \phi_s(x)|\phi_s(x)\rangle = \frac{it^*}{2} \langle \phi_s(x)|\phi_s(x)\rangle - \langle \phi_s(x)|\,\widehat{x\tfrac{d}{dx}}\,|\phi_s(x)\rangle. \tag{80}$$

By using Eq. (74) in Eq. (80), we again obtain the expression in Eq. (76), i.e.,

$$\sigma = \frac{4}{\langle \phi_s(x)|\phi_s(x)\rangle} \int_1^\infty \phi_s(x) x \frac{d}{dx}\phi_s(x) dx + \frac{5}{2}, \tag{81}$$

with the ansatz given by Eq. (7). From evaluating Eq. (81) at $x \geq 1$ owing to the difference operator $\hat{\Delta}\Psi_s(x)$, it can be seen that

$$\begin{aligned} \Re(\sigma) &= \Re\Big[\frac{4}{\langle \phi_s(x)|\phi_s(x)\rangle} \int_1^\infty \phi_s(x) x \frac{d}{dx}\phi_s(x) dx\Big] + \frac{5}{2} \\ &= \frac{1}{2}. \end{aligned} \tag{82}$$

Remark 2 *The Riemann Hypothesis states that the real part of all of the nontrivial zeros of the Riemann zeta function are located at $\sigma = 1/2$ [8].*

Remark 3 *Solutions to Eq. (3) are symmetric about the origin, i.e., $x \in [1, \infty)$, $[-1, -\infty)$, and subject to the singularity at $\phi_s(x = 0) = 0$ [25].*

Lemma 1 Substituting $s_n = 1/2 - it_n$ $(n = 1, 2, 3, \dots)$ in the reflection formula

$$\zeta(s_n) = 2^{s_n}\pi^{s_n - 1} \sin\Big(\frac{\pi s_n}{2}\Big)\Gamma(1 - s_n)\zeta(1 - s_n) \tag{83}$$

shows that all of the *nontrivial* zeros of the zeta function Eq. (1) occur at $\Re(\sigma) = 1/2$ when $t_n = 1 - 2n$.

HILBERT-PÓLYA CONJECTURE

The Hilbert-Pólya conjecture states that there exists a hermitian operator in an infinite-dimensional Hilbert space whose eigenvalues are the nontrivial Riemann zeros [18].

Theorem 6 The operator $\hat{H} = -i\sqrt{x}\partial_x\sqrt{x}$ satisfies the Hilbert-Pólya conjecture at the normalization constraint $x = 1$.

Proof 6 *Since $\partial_s = 1/2(\partial_\sigma - i\partial_t)$,*

$$-2\sqrt{x}\partial_x\sqrt{x} \longmapsto i\sqrt{x}\partial_x\sqrt{x} \tag{84}$$

in Eq. (27). The eigenvectors of $\hat{H} = -i\sqrt{x}\partial_x\sqrt{x}$ with eigenvalues $t_n = 1 - 2n$ corresponding to functions $\psi_n(x)$ are given from

$$i\sqrt{x}\partial_x\sqrt{x}\psi_n(x) = (1 - 2n)\psi_n(x) \tag{85}$$

as

$$\psi_n(x) = \frac{1}{x^{1/2+i(1-2n)}}. \tag{86}$$

Alternatively,

$$-2\sqrt{x}\partial_x\sqrt{x}\frac{1}{x^\sigma} = \frac{\sigma - 1/2}{x^\sigma} \tag{87}$$

since $1 - 2n = i(\sigma - 1/2)$. *Moreover,*

$$-2\left(\frac{1}{2} - \sigma\right)x^{-\sigma} = \left(\sigma - \frac{1}{2}\right)x^{-\sigma} \tag{88}$$

such that $\sigma = 1/2$ *at* $x = 1$.

CONCLUSION

In this study, we have discussed the convergence of the real part of every nontrivial zero of the analytic continuation of the Riemann zeta function. This was accomplished by developing a Riemann zeta Schrödinger equation and comparing it with both the Bender-Brody-Müller and the Hilbert-Pólya conjecture in both configuration space and momentum space. A symmetrization procedure was implemented to study the convergence of the system, and the expectation values were calculated from the resulting system to study the nontrivial zeros of the analytic continuation of the Riemann zeta function. Moreover, a second quantization procedure was performed for the Riemann zeta Schrödinger equation to obtain the equations of motion. Finally, a normalized convergent expression for the analytic continuation of the nontrivial zeros of the Riemann zeta function was obtained, and a convergence test for the expression was performed heuristically demonstrating that the real part of every nontrivial zero of the Riemann zeta function converges at $\sigma = 1/2$.

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