SWIMMING AT LOW REYNOLDS NUMBERS

Consider the motion of sperm at low Reynolds number. Sperm is modeled as a sphere (head) attached to the slender filament (tail). Thrust generated by the flagellar motion of the tail balances the drag on the head and pushes the sperm in the fluid. At low Reynolds number, inertial terms are neglected and as a first approximation, for incompressible Newtonian fluid at steady state, we get

\[ \nabla p = \mu \nabla^2 \vec{u} \]
\[ \nabla \cdot \vec{u} = 0 \]

Flow past sphere (drag calculation):
Consider a sphere of radius \( a \) placed in a uniform stream of velocity \( U \) (in case of moving sphere, we change the frame of reference). Flow is axisymmetric and only the azimuthal component of vorticity is non-zero

\[ \omega_s = \frac{1}{r} \left[ \frac{\partial (ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right] \]

Define a stream function (Incompressibility condition ensures the existence of such a function) as follows:

\[ u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} ; u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \]
\[ \therefore \nabla^2 \omega_\psi = 0 \]
\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \psi = 0 \]

Boundary conditions are

\[ u_r = 0 \quad \text{at} \ r = a \quad \Rightarrow \psi(a, \theta) = 0 \]
\[ u_\theta = 0 \quad \text{at} \ r = a \quad \Rightarrow \frac{\partial}{\partial r} \psi(a, \theta) = 0 \]

Uniform farfield \( @ r \to \infty \Rightarrow \psi(\infty, \theta) = \frac{1}{2} Ur^2 \sin^2 \theta \)

Farfield b.c. is nothing but the stream function for uniform flow in spherical coordinates. Using separation of variables while noting the function factors respect the farfield conditions, we use

\[ \psi = f(r) \sin^2 \theta \]
\[ \therefore \frac{d^4 f}{dr^4} - \frac{4}{r^2} \frac{d^2 f}{dr^2} + \frac{8}{r^3} \frac{df}{dr} - \frac{8f}{r^4} = 0 \]

Whose solution is

\[ f = Ar^4 + Br^2 + Cr + \frac{D}{r} \]
\[ A = 0, B = U/2, C = -3Ua/4, D = Ua^3/4 \]

Thus, the velocity components are
Using this field in the momentum equation, pressure is solved as
\[ p = -\frac{3a\mu U \cos \theta}{2r^2} \]
To calculate the drag on sphere surface, one can integrate the viscous stress over the surface. The component of the drag force per unit area in the direction of uniform stream is
\[ (-p \cos \theta + \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta)_{r=a} \]
\[ \sigma_{rr} = 2\mu \frac{\partial u_r}{\partial r} = 2\mu U \cos \theta \left( \frac{3a}{2r^2} - \frac{3a^3}{2r^4} \right)_{r=a} = 0 \]
\[ \sigma_{r\theta} = \mu \left[ r \frac{\partial}{\partial r} \left( \frac{u_r}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] = \frac{3\mu U a^3 \sin \theta}{2r^4} \bigg|_{r=a} = \frac{3\mu U \sin \theta}{2a} \]
Therefore drag is
\[ D = (4\pi a^2) \frac{3\mu U}{2a} = 6\pi \mu a U \]
It was noted that the assumption that inertial terms are negligible as compared to viscous terms is not valid in the far field and Oseen’s correction was introduced to account for these effects.

**Doublet (Dipole)**
For potential flows, velocity field can be represented as the gradient of a scalar function. Note that the existence of the potential field implies the irrotational condition. The solution of a dipole of strength \( G \) (direction from “negative to positive” or “sink to source”) is
\[ \phi = \vec{\nabla} \cdot \left( \frac{\vec{G}}{4\pi r} \right); \vec{u} = \vec{\nabla} \left( \frac{\vec{G}}{4\pi r} \right) \]
Note that velocity falls off rapidly as inverse-cube law in the far field and on a particular spherical surface \( r=a \), the normal component of velocity field would match a given constant vector if the following relation between the constant vector and the dipole strength is satisfied (stated without proof)
\[ \vec{G} = 2\pi a^3 \vec{U} \]

**Stokeslet**
To represent the effect of a concentrated external force \( F \) acting on a single point of the fluid, a modified force-balance equation must be solved. A dirac delta-distribution of force per unit volume is used and the position vector now represents the vector displacement from the point of application of the force.
\[ \vec{F} \delta(\vec{r}) - \vec{\nabla} p + \mu \nabla^2 \vec{u} = 0 \]
Taking the divergence of the above equation and using the continuity equation, we get the following equation for pressure:
\[ \nabla^2 p = \nabla \cdot \left( \vec{F} \delta(\vec{r}) \right) \]

\[ \therefore p = \nabla \cdot \left( -\frac{\vec{F}}{4\pi r} \right) \]

The stokeslet velocity distribution can be written as (stated without proof)

\[ \vec{u} = \frac{r^2 \vec{F} + (\vec{F} \cdot \vec{r}) \vec{r}}{8\pi \mu r^3} \]

The same velocity field can be written as

\[ \vec{u} = \frac{\vec{F}}{6\pi \mu r} + \frac{r^2}{4} \left( \nabla \cdot \left( \frac{\vec{F}}{6\pi \mu r} \right) \right) \]

Note that on a spherical surface \( r=a \), the second term can be cancelled by a dipole velocity field of strength

\[ \vec{G} = -\frac{a^2 \vec{F}}{6\mu} \]

The resulting velocity then takes the following value at \( r=a \),

\[ \vec{u} = \frac{\vec{F}}{6\pi \mu a} \]

Also, the stokeslet velocity field contains second order surface harmonics (Solution techniques and their analysis is deferred till the astrophysical fluid flow problems)

Flagellum can be represented by the parametric curve \((X(s), Y(s), Z(s))\), where \(s\) is the arc length along the flagellum.

*Analysis of flagellum (stated without proof)*

If \( f(s) \) is the force per unit length with which a flagellum of small radius \( a \) acts on the fluid, then the resulting fluid motion can be represented by a distribution of stokeslets along the centerline, of strength \( f(s) \) per unit length, accompanied by dipoles of strength

\[ -\frac{a^2 f_n(s)}{4\mu} \]

per unit length, where \( f_n(s) \) is the vector normal to the centerline obtained by resolving \( f(s) \) onto the plane normal to centerline.

*Implications on sperm motion*

Balance the thrust generated by the tail through the flagellar motion with the drag experienced by the head using Stokes’ drag law.

*Homework assigned:* N/A (verify some of the algebra to get the intermediate steps of the various results)
Flagellar motor (Ref: DeRosier1998, Berry 2001)

Bacteria Flagellum

(a) Schematic of the bacterial flagellum as a mechanical device. The parts thought to be responsible for torque generation are shown in thicker lines. They lie inside the cell except for the studs, which cross the membrane and are thought to be anchored to the peptidoglycan layer (PG). The filament or propeller lies outside the cell’s inner (IM) and outer (OM) membranes and is about 10 μm long.

(b) Image of the flagellum. The image is an average of about 100 individual electron micrographic images of a frozen-hydrated preparation of flagella. The bar corresponds to 25 nm.

(c) A color-coded copy of a slice through the cylindrically averaged flagellar structure. FIPv, which forms the M ring, S ring, and socket, is shown in green. Flig, shown in blue, is attached to the cytoplasmic face of the M ring. FlIM and FlIN, shown in yellow, are thought to make up the C ring. The L and P rings are shown in pink, and the rod and the short stretch of hook are gray.
Filaments are made of superparamagnetic 1 μm diameter streptavidin coated particles linked by several 107 nm long double-stranded DNA with biotin.

This filament is attached to a red blood cell (RBC) to create a micro-swimmer with a head and magnetic tail.

A combination of longitudinal and time-periodic transverse magnetic field creates a beating pattern in the filament that can propel this micro-swimmer.

References: