FLOW THROUGH POROUS MEDIA

Biological tissues or membranes can be modeled like porous media. Flow through porous media has been extensively studied by geologists as well as civil/petroleum engineers. The governing equations for flow through simple homogeneous porous solid relate the average flow velocity to the applied pressure gradient and are known as Darcy’s Law.

Porous Media:
Porosity (specifically volumetric) is defined as the ratio of pore volume to the total volume (pore + grain).

Void fraction is the ratio of pore volume to the volume of the solid grains.

Figure 1: Schematic representation of porous media to show various length scales associated with pore, grain and the macroscopic flow regime.

Macroscopic scales: length scales large ($L$ in fig. 1) as compared to the grain or pore size of the media. Flow is seen as continuous phenomena in space at this scale.

Microscopic scales: length scale commensurate with the grain or pore size ($l_p$ or $l_g$ in fig. 1) of the media, but still large as compared to the molecular movements/dimensions (fluid is continuum).

Value of a macroscopic quantity at a given point needs to be defined on a representative elemental volume (REV) such that flow is continuous as well as media is a continuum (pore as well as grain features are averaged out in REV).
Similitude:
1. Geometric similarity: Two systems are said to be geometrically similar if for every pair of length scales, their ratio is constant.
2. Kinematic similarity: Two systems are said to be kinematically similar if for every point in the flow field the velocity vectors in corresponding system have same directions and their magnitude ratio is constant.
3. Dynamic similarity: Two systems are said to be dynamically similar if for every point in the flow field the components of forces on the fluid element have the same directions in both systems and their magnitude ratio is constant.

Derivation of Darcy’s law from N-S:
Consider the steady Navier-Stokes equation as the governing equation for the fluid in the pores.

\[ \rho u_j \frac{\partial u_i}{\partial x_j} = \mu \frac{\partial^2 u_i}{\partial x^2_j} + \rho \left( f_i^b - \frac{1}{\rho} \frac{\partial p}{\partial x_i} \right) \]

Here, \( f_i^b \) is the volumetric body force (such as gravity). One could express the above-stated equation as the balance between advective forces, viscous forces and the driving forces.

In transforming averaged contributions of each force term from microscopic level to macroscopic level, we carry out the averaging of individual terms at REV scale.
Macroscopic balance of driving force terms simply equates to the averaged driving forces over the pore volume.

For the system under consideration, there is no macroscopic gain in velocity at REV scale and therefore, the advective forces averaged over REV from microscopic scale contributions should be zero. In other words, the steady REV-scale averaged velocity does not change in space.

Contribution of the viscous terms from microscopic scale to macroscopic scale also simplifies by noting the fact that only the terms along the macroscopic flow direction can contribute (other terms should yield zero because there is no net flow in those directions). Using the similitude arguments, we can advocate that the viscous terms should behave as follows

\[
\frac{\partial^2 u_j}{\partial x_j^2} = \alpha \frac{u_j}{l_p^2}
\]

Here, \( \alpha \) is the proportionality constant for the REV (can be different for different REVs).

Average value of \( \alpha \) over REV is written as \( 1/N \). Substituting these results in the balance, yields the following relation:

\[
u_s = N l_p^2 \frac{\rho}{\mu} \left[ f_i^b - \frac{\nabla p}{\rho} \right]
\]

\[
\Rightarrow \quad u_s = \frac{K}{\mu} \left[ f_i^b - \frac{\nabla p}{\rho} \right]
\]

The last relationship is known as Darcy’s Law for flow through porous media.

**Homework assignment: (Due in one week from the date of handing out)**

Derive the relationship between superficial velocity and pressure gradient for a capillary model of the porous media using the solution for laminar flow field in a pipe (Poiseuille flow).

**References:**