PERISTALTIC PUMPING

Peristaltic pumping: Phenomena in which tubular structures propel their contents by organized contractions of longitudinal and circular muscle fibers. Such contractions lead to localized reductions in the diameter of the tubes that organize into waves propagating along the tube axis. Together with applied pressure differences across the tube, these waves can result in the net transport of fluid within the tube.

Examples: Transport in esophagus, intestines, ureter and fallopian tubes.

Analysis:
Consider cylindrical coordinate systems for the fixed frame \((R,Z)\) and the moving frame \((r,z)\). The radial and transverse velocities in these two frames are denoted \((U_r,U_z)\) and \((u_r,u_z)\) respectively.

\[
\begin{align*}
    z &= Z - ct, \\
    r &= R \\
    u_z(r,z) &= U_z(R,Z - ct) - c, \\
    u_r(r,z) &= U_r(R,Z - ct).
\end{align*}
\]

Figure 1. Schematic of peristaltic wave in cylindrical coordinates

Simplifying assumptions:
Characteristic length scales in the peristaltic pumping are: 1) \(a\), the radius of the circular tube, 2) \(b\), the magnitude of the wave amplitude and 3) \(\lambda\), the wavelength. It will be assumed that the wavelength is much larger than tube radius i.e. \(\lambda >> a\). Further, the wave amplitude can not exceed the tube radius i.e. \(b << a\).
Simplified governing equations and boundary conditions:

Wave train:

\[
h(z, t) = a + b \sin \left( \frac{2\pi}{\lambda} (z - ct) \right)
\]

Parabolic velocity profile:

\[
\frac{u_z}{c} = -\frac{a^2}{4\mu c} \frac{dp}{dz} \left[ \left( \frac{h}{a} \right)^2 - \left( \frac{r}{a} \right)^2 \right] - 1.
\]

Flow rate:

\[
q = \int_0^h 2\pi u_z dr
\]

\[
q = \pi a^2 c \left[ -\frac{h^4}{8\mu c^2} \frac{dp}{dz} \left( \frac{h}{a} \right)^2 \right]
\]

Lab frame:

\[
Q = \int_0^h 2\pi R U_z dR
\]

\[
Q = q + \pi h^2 c
\]

Time-averaged flow rate:

\[
\overline{Q} = \frac{1}{T} \int_0^T Q dt
\]

\[
\overline{Q} = q + \pi a^2 c \left( 1 + \frac{1}{2} \left( \frac{b}{a} \right)^2 \right)
\]

Define \( \phi = \frac{b}{a} \), to get \( \overline{Q} = q + \pi a^2 c \left( 1 + \frac{1}{2} \phi^2 \right) \)
Pressure rise per wavelength:

\[ \Delta p_\lambda = \int_0^\lambda dp \frac{dz}{dz} \]

\[ = -8\mu \int_0^\lambda \frac{dz}{h^2} - \frac{8\mu q^2}{\pi} \int_0^\lambda \frac{dz}{h^4} \]

\[ = \frac{4\mu c \lambda}{a^2} \left\{ 8\phi^2 \left(1 - \frac{\phi^2}{16}\right) - 4\phi \left(1 - \frac{\phi}{4}\right) \left(1 + \frac{3\phi^2}{2}\right) \right\} \]

Where

\[ \Theta = \frac{\bar{Q}}{\pi a^2 c \left(2\phi - \frac{\phi^2}{2}\right)} \]

**Homework assignment (Due one week from the date of handing out):**

Rework analysis of peristaltic pumping for a two-dimensional flow in a channel of mean height 2a. Obtain the expression for axial component of velocity and its time average. State all the assumptions clearly and then simplify the full Navier-Stokes equations to this particular case.

**References:**