Operational approach to phase-space measurements in quantum mechanics

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Wódkiewicz has derived an operational formula for a positive phase-space distribution function in quantum mechanics (see also ref. 2). Here we point out that the proposed formula is actually a special case of a two-particle Wigner distribution function in which correlations have been neglected. We present a new operational formula which includes correlations. Also, we incorporate a brief description of the role and significance of quantum distribution functions in general.

Realizing that one of the most powerful approaches to classical mechanics is through the concept of phase-space, Wigner introduced a quantum distribution function of position and momentum coordinates, which provides a framework for an exact reformulation of non-relativistic quantum mechanics in terms of classical concepts. A generalization to the case of spin one half particles has also been presented.

In particular, the use of Wigner's distribution function permits one to represent a quantum-mechanical ensemble average by a classical phase-space integral. Also, as distinct from Schrödinger-Heisenberg quantum mechanics, a development of various results in powers of \( \hbar \) is relatively straightforward, the limit \( \hbar \to 0 \) leading immediately to classical mechanics. Thus, it is not only a useful calculational tool but also provides insights into the nature of quantum mechanics and its role in such areas as measurement theory. Similar remarks apply to other quantum distribution functions which are in common use in many areas of non-equilibrium statistical mechanics, most particularly in quantum optics.

These functions and their inter-relationships are reviewed in ref. 6.

Wódkiewicz defines his function thus:

\[
P(q, p) = \int dq' dp' W(q, q', p, p') W(q, p')
\]  

(1)

where \( q, q' \) and \( p, p' \) refer to position and momentum coordinates, respectively, and \( W \) denotes the Wigner quantum distribution function, \( W \) describing the state of a laser pulse with which a detected particle interacts and \( W \) describing the state of the particle after the interaction. Note that \( P(q, p) \) is always positive (in contrast to \( W(q, p) \) which can sometimes be negative), and that the so-called "... deficiencies of Wigner's calculation..." have been explained.

Our main purpose here is to point out that \( P(q, p) \) is not a new phase-space distribution function. It is, in fact, a special case of a two-particle Wigner distribution function in which correlations have been neglected. (Here 'particle' should be interpreted in the general sense of incorporating the systems \( \phi \) and \( \psi \) referred to above.) The \( W \) functions appearing in equation (1) are single-particle Wigner functions and these are the kind most often discussed. However, Wigner's original paper actually introduced the general \( n \)-particle function, \( W^{(n)} \), whose properties are discussed in ref. 7. In particular, these functions obey a quantum BRGKY (Hopf-Kohn-Young-Kirkwood-Young) hierarchy of equations, the rate of change of \( W^{(n)} \) depending not only on \( W^{(n)} \) but also on the higher-order function \( W^{(k)} \).

The two-particle Wigner function for a pure state, \( \psi^{(2)} \), may be written in the form (equation 5 of ref. 2, with \( n = 2 \))

\[
W^{(2)}(q_1, q_2; p_1, p_2) = \int dq_3 dp_3 \psi^* (q_3 + \frac{1}{2}q_1; q_2, q_3, p_3) \exp \left[ \frac{ip_3(q_3 - q_1)}{\hbar} \right]
\]  

(2)

where \( \psi \) is the wave function for the system. If we now assume that there are no correlations between particles 1 and 2, we may write

\[
0(1, 2) = 0(1)0(2)
\]  

(3)

from which it follows that

\[
W^{(2)}(q_1, q_2; p_1, p_2) = W(q_1, p_1)W(q_2, p_2)
\]  

(4)

that is the two-particle function may be written as a product of single-particle functions. Hence, from equations (1) and (4), we deduce that

\[
P(q, p) = \int dq' dp' W^{(2)}(q + q', p + p', p')
\]  

(5)

In other words, the operational phase-space distribution function \( P(q, p) \) is obtained by starting with the two-particle Wigner function \( W^{(2)}(q_1, q_2; p_1, p_2) \) and then (1) assuming that there is no correlation between particles 1 and 2; (2) that \( q_1 = q_2 = q' \); and \( p_1 = p + p' = p_2 \); and (3) integrating over all \( q' \) and \( p' \).

We conclude that \( P(q, p) \), as defined in equation (1), is a special case of a two-particle Wigner distribution function. In addition, we may regard \( P(q, p) \), as defined in equation (5), as a new operational formula for a distribution function, which includes correlations.

We are not decrying the work of Wódkiewicz—which has the merit of deriving \( P(q, p) \) from a dynamical process involving the detected particle, the detector, and the filtering device—but we are rather putting it in its proper context. In essence, it does not highlight "... deficiencies of Wigner's calculation..." but rather it displays the wide scope and power of the results presented by Wigner.

In the same context, we point out that Wódkiewicz's function \( P(q, p) \) is the same as the function \( J(q, p) \) introduced by R.F.O'C. and Rajagopal, and, in their demonstration of the fact that the "... new interpretation of the scalar product in Hilbert space..." by Aharonov et al. is essentially equivalent to Wigner's exact reformulation of non-relativistic quantum mechanics using distribution functions. R.F.O'C. and Rajagopal also demonstrated explicitly that \( P(q, p) = J(q, p) \) (equation 6 of ref. 7) but it is perhaps useful to point out that such a result is also contained in the earlier work of R.F.O'C. and Wigner (1) (see their equations (2) and (5)), which used such a result to prove that the Husimi distribution (15) (which is obtained by smoothing a Wigner distribution function with the Wigner distribution function for the ground-state of a harmonic oscillator) is always positive. The Aharonov et al. interpretation can also be regarded as the one underlying the stochastic phase-space formulation of non-relativistic quantum mechanics, as demonstrated by Prugovečki.

All previous discussions treated pure states, but a generalization to mixed states is readily obtained. In fact, using the general result (equation (1.21) of ref. 7) that

\[
\int dq dp A(q, p)B(q, p) = \langle 2\hbar | V | A\tilde{B}\rangle
\]  

(6)

where \( \tilde{A} \) denotes the trace and \( \tilde{A} \) and \( \tilde{B} \) are operators in Hilbert space and where \( A(q, p) \) and \( B(q, p) \) are the corresponding Wigner phase-space functions, and also using the fact that the phase-space function corresponding to the density matrix \( \tilde{\rho} \) is \( (2\hbar)N \) times the distribution function, it follows that (taking \( \tilde{A} = A_0 \) and \( \tilde{B} = B_0 \)).
\[ \int d\mathbf{q} d\mathbf{p} \, W_\alpha(\mathbf{q}, \mathbf{p}) W_\beta(\mathbf{q}, \mathbf{p}) - (2\pi)^{-1} Tr(\hat{\rho}_\alpha \hat{\rho}_\beta) > 0 \quad (7) \]

a result also reported by Isgur on the S-matrix description of particle states and of their measurements. In other words, the operational phase-space distribution function in the absence of correlations is always non-negative, even for mixed states.

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