The electromagnetic field appears almost everywhere in physics. Following the introduction of Maxwell’s equations in 1864, Max Planck initiated quantum theory when he discovered $h = 2\pi\hbar$ in the laws of black-body radiation. In 1905 Albert Einstein explained the photoelectric effect on the hypothesis of a corpuscular nature of radiation and in 1917 this paradigm led to a description of the interaction between atoms and electromagnetic radiation.

The study of quantized field effects requires an understanding of the quantization of the field which leads to the concept of a quantum of radiation, the photon. Specific nonclassical features arise when the field is prepared in particular quantum states, such as squeezed states. When the radiation field interacts with an atom, there is an important difference between a classical field and a quantized field. A classical field can have zero amplitude, in which case it does not interact with the atom. On the other hand, a quantized field always interacts with the atom, even if all the field modes are in their ground states, due to vacuum fluctuations. These lead to various effects such as spontaneous emission and the Lamb shift.

The interaction of an atom with the many modes of the radiation field can conveniently be described in an approximate manner by a master equation where the radiation field is treated as a reservoir. Such a treatment gives a microscopic and quantum mechanically consistent description of damping.

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### References
78.1 Field Quantization

This section provides the basis for the quantized field effects discussed in this Chapter [78.1]. We expand the field in a complete set of normal modes which reduces the problem of field quantization to the quantization of a one dimensional harmonic oscillator corresponding to each normal mode.

The classical free electromagnetic field, i.e., the field in a region without charge and current densities, obeys the Maxwell equations

\[ \nabla \cdot B = 0 , \]
\[ \nabla \cdot D = 0 , \]
\[ \nabla \times E + \frac{\partial B}{\partial t} = 0 , \]
\[ \nabla \times H - \frac{\partial D}{\partial t} = 0 , \]

where \( B = \mu_0 H, D = \varepsilon_0 E. \) The magnetic permeability \( \mu_0 \) connects the magnetic induction \( B \) with the magnetic field \( H \) and the electric permittivity \( \varepsilon_0 \) of free space connects the displacement \( D \) with the electric field \( E \). In the case of a free field, \( E \) and \( B \) may be obtained from

\[ B = \nabla \times A, \]
\[ E = -\frac{\partial A}{\partial t}, \]

where the vector potential \( A \) obeys the Coulomb gauge condition \( \nabla \cdot A = 0 \) and satisfies a wave equation. In order to solve this wave equation we expand the vector potential

\[ A(x, t) = \sum_{k, \sigma} \left( \frac{\hbar}{2 \omega_0 \varepsilon_0 V} \right)^{1/2} \]
\[ \times \left[ \alpha_{k\sigma} \epsilon_{k\sigma} e^{i(k \cdot x - \omega_0 t)} + \text{c.c.} \right] \]

in a set of normal modes \( V^{-1/2} \exp(ik \cdot x)\epsilon_{k\sigma} \) which are orthonormal in the volume \( V \). Due to the gauge condition \( \nabla \cdot A = 0 \), we obtain two orthogonal polarization vectors \( \epsilon_{k1} \) and \( \epsilon_{k2} \) with \( \epsilon_{k1} \cdot k = 0 \) for each wave vector \( k \). The dispersion relation is \( \omega_k = c|k| \). The Fourier amplitudes \( \alpha_{k\sigma} \) are complex numbers in the classical theory.

The field is quantized by replacing the classical amplitude \( \alpha_{k\sigma} \) by the mode annihilation operator \( a_{k\sigma} \). The complex conjugate \( \alpha_{k\sigma}^* \) is replaced by the mode creation operator \( a_{k\sigma}^\dagger \). They obey the commutation relation

\[ [a_{k\sigma}, a_{k'\sigma'}^\dagger] = \delta_{kk'} \delta_{\sigma\sigma'} . \] (78.8)

The representation of the electric field operator

\[ E(x, t) = i \sum_{k, \sigma} \left( \frac{\hbar \omega_k}{2 \varepsilon_0 V} \right)^{1/2} \]
\[ \times \left[ a_{k\sigma} \epsilon_{k\sigma} e^{i(k \cdot x - \omega_0 t)} - \text{h.c.} \right] \]
\[ \equiv E^+(x, t) + E^-(x, t) \] (78.9)
in terms of these operators follows from (78.6) and the operator for the vector potential. Note also the often used decomposition of the electric field operator into the positive and negative frequency parts \( E^+ \) and \( E^- \) respectively. A similar relation holds for the operator describing the magnetic induction \( B \).

Using the operators for the electric and magnetic field, one can transform the field energy

\[ H = \frac{1}{2} \int dV \left( \varepsilon_0 E^2 + B^2 / \mu_0 \right) \]

into the form

\[ H = \sum_{k, \sigma} \hbar \omega_k \left( a_{k\sigma}^\dagger a_{k\sigma} + 1/2 \right) , \]

which is a sum of independent harmonic oscillator Hamiltonians corresponding to each mode \( (k, \sigma) \). The number operator \( N_{k\sigma} = a_{k\sigma}^\dagger a_{k\sigma} \) represents the number of photons in the mode \( (k, \sigma) \), while \( \hbar \omega_k / 2 \) is the energy of the vacuum fluctuations.

Hence each mode of the electromagnetic field is equivalent to a harmonic oscillator. In the next section we discuss specific states of a single mode. The general quantum state of the electromagnetic field consisting of many modes is given by a superposition of product states that are composed out of these single mode states.

78.2 Field States

This section summarizes the properties of several important states of the electromagnetic field. From the independence of the normal modes, the discussion may be restricted to a single normal mode. With the mode index \( (k, \sigma) \) suppressed, a single mode Hamiltonian is

\[ H = \hbar \omega \left( a^\dagger a + 1/2 \right) = \frac{\hbar \omega}{2} p^2 + \frac{\hbar \omega}{2} x^2 . \] (78.12)
In the second step, the quadrature operators

\[ x = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad (78.13) \]
\[ p = \frac{1}{1\sqrt{2}} (a - a^\dagger), \quad (78.14) \]

are introduced, which are equivalent to scaled position and momentum operators of a massive particle in a harmonic potential. The quadratures of a quantized field are measurable with the help of homodyne detection as discussed in Sect. 78.4.1.

We shall now describe several states of this quantized field mode: number states, coherent states, squeezed states, Schrödinger cats, and phase states. A quantized field in a coherent state shows the most classical behavior. A superposition of two coherent states, which is a Schrödinger cat, already shows nonclassical features. Number states and squeezed states are further typical examples of nonclassical states.

78.2.1 Number States

The eigenstates of the Hamiltonian (78.12) are the eigenstates of the number operator \( N = a^\dagger a \),

\[ N|n\rangle = n|n\rangle, \quad (78.15) \]

where \( n = 0, 1, 2, \ldots \) denotes the excitations or the number of photons in the mode. The vacuum state of the mode \( |0\rangle \), is defined by

\[ a|0\rangle = 0. \quad (78.16) \]

The ladder of excitations can be climbed up and down via the application of creation and annihilation operators

\[ a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \quad (78.17) \]
\[ a|n\rangle = \sqrt{n}|n-1\rangle, \quad (78.18) \]

on a Fock state \( |n\rangle \). These number or Fock states form a complete and orthonormal set of states so that

\[ \sum_{n=0}^\infty |n\rangle\langle n| = 1, \quad \langle n|k\rangle = \delta_{nk}. \quad (78.19) \]

Their quadrature representations are

\[ \langle x|n\rangle = (\sqrt{2^n n!})^{-1/2} H_n(x) e^{-x^2/2}, \quad (78.20) \]
\[ \langle p|n\rangle = (\sqrt{2^n n!})^{-1/2} (-i)^n H_n(p) e^{-p^2/2}. \quad (78.21) \]

The states \( |x\rangle \) and \( |p\rangle \) are eigenstates of the quadrature operators \( x \) and \( p \), (78.13) and (78.14).

Number states provide a frequently used representation of a pure quantum state

\[ |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad (78.22) \]

or a mixed quantum state given by the density operator

\[ \rho = \sum_{n,k} \rho_{nk} |n\rangle \langle k| \quad (78.23) \]

(Chapt. 7).

78.2.2 Coherent States

The coherent state is a specific superposition of number states. In contrast to a number state, a coherent state does not possess a definite number of photons: the photon distribution is Poissonian. For a large average photon number, the electric and magnetic fields have rather well defined amplitudes and phases with vanishing relative quantum fluctuations. Hence the Poissonian photon distribution frequently serves as a borderline between classical and nonclassical field states. Nonclassical states show a sub-Poissonian behavior. An extreme example is a field prepared in a number state. A parameter which quantifies the deviations from Poissonian behavior is the \( Q \) parameter introduced by Mandel [78.2].

We define the coherent state \( |\alpha\rangle \) as an eigenstate of the annihilation operator

\[ a|\alpha\rangle = \alpha|\alpha\rangle \quad (78.24) \]

with the complex amplitude \( \alpha = |\alpha|e^{i\theta} \). The coherent state can be represented by

\[ |\alpha\rangle = e^{(-\alpha^*a^\dagger + \alpha a)/2} |0\rangle = D(\alpha)|0\rangle, \quad (78.25) \]

that is, by the action of the displacement operator \( D(\alpha) \) on the vacuum.

The number state representation of \( |\alpha\rangle \) reads

\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (78.26) \]

A coherent state \( |\alpha_0\rangle \) that evolves in time according to the free field Hamiltonian (78.12) stays coherent, i.e.,

\[ |\psi(t)\rangle = \exp (-iHt/\hbar) |\alpha_0\rangle = e^{-i\alpha_0t/2} |\alpha(t)\rangle, \quad (78.27) \]

with amplitude \( \alpha(t) = \alpha_0 \exp[-i\omega t] \).
Another important representation of a coherent state is the $x$ representation

$$\langle x | \alpha \rangle = \pi^{-1/4} \exp \left\{ -|\text{Re}(\alpha)|^2 \right\} \times \exp \left( -x^2/2 + \sqrt{2}ax \right),$$  
(78.28)

where $|x\rangle$ denotes again the $x$ quadrature eigenstate.

The photon distribution in a coherent state

$$|\langle n | \alpha \rangle|^2 = |\alpha|^2 e^{-|\alpha|^2}/n!$$  
(78.29)

is a Poisson distribution with average photon number $\langle N \rangle = |\alpha|^2$ and variance $(\Delta N)^2 = (N^2) - \langle N \rangle^2 = |\alpha|^2$. Hence the relative fluctuations $(\Delta N)/\langle N \rangle = (N - \langle N \rangle)^2/\langle N \rangle$ vanish for a large average photon number.

The Mandel $Q$ parameter

$$Q = \frac{\langle \Delta N \rangle^2 - \langle N \rangle}{\langle N \rangle}$$  
(78.30)

vanishes for a field in a coherent state. A nonclassical field may show sub-Poissonian behavior with $Q < 0$. As an example, the Schrödinger cat state is a superposition of two coherent states

$$|\text{cat}\rangle = \left( 2 + 2 \cos (\alpha^2 \sin \phi) e^{-\alpha^2 \sin^2 \phi/2} \right)^{-1/2} \times \left[ |\alpha e^{i\phi/2}\rangle + |\alpha e^{-i\phi/2}\rangle \right],$$  
(78.31)

where $\alpha$ is assumed to be real. The $Q$ parameter for this superposition state, shown in Fig. 78.1, takes on negative values for specific angles $\phi$. The nonclassical behavior of such a $|\text{cat}\rangle$-state can be explained [78.3] as a result of quantum interference between the two coherent states present in (78.31). The incoherent superposition described by the density operator

$$\rho = \frac{1}{2} \left( |\alpha e^{i\phi/2}\rangle \langle \alpha e^{i\phi/2}| + |\alpha e^{-i\phi/2}\rangle \langle \alpha e^{-i\phi/2}| \right)$$  
(78.32)

does not have this nonclassical character: its $Q$ parameter vanishes. Coherent states have a direct physical significance: the quantum state of a stabilized laser operating well above threshold can be approximated by a coherent state.

### 78.2.3 Squeezed States

Squeezed states [78.4–6] minimize the uncertainty product of the quadrature components of the electromagnetic field. The quadrature components $x$ and $p$ of the single mode field are defined in (78.13) and (78.14). They obey the commutation relation $[x, p] = i$. Their uncertainties $(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 \equiv \langle p^2 \rangle - \langle p \rangle^2$ fulfill the Heisenberg inequality

$$\Delta x \Delta p \geq 1/2.$$  
(78.33)

The coherent state is a special minimum uncertainty state with equal uncertainties $\Delta x = \Delta p = 1/\sqrt{2}$. Squeezed states comprise a more general class of minimum uncertainty states with reduced uncertainty in one quadrature at the expense of increased uncertainty in the other. These states $|\alpha, \epsilon\rangle$ are obtained by applying the displacement operator $D(\alpha)$ and the unitary squeeze operator $S(\epsilon)$ to the vacuum

$$|\alpha, \epsilon\rangle = D(\alpha) S(\epsilon)|0\rangle.$$  
(78.35)

The squeeze operator $S(\epsilon)$ transforms $a$ and $a^\dagger$ according to

$$S(\epsilon)^\dagger a^\dagger S(\epsilon) = a \cosh \epsilon - a^\dagger e^{-2i\epsilon} \sinh \epsilon,$$  
(78.36)

$$S(\epsilon^\dagger) a S(\epsilon) = a^\dagger \cosh \epsilon - a e^{2i\epsilon} \sinh \epsilon,$$  
(78.37)

where $\epsilon = r e^{-2i\phi}$. The rotated quadratures

$$X_1 = x \cos \phi - p \sin \phi,$$  
(78.38)

$$X_2 = p \cos \phi + x \sin \phi,$$  
(78.39)

transform according to

$$S(\epsilon)(X_1 + iX_2) S(\epsilon^\dagger) = X_1 e^{-\epsilon} + iX_2 e^{\epsilon},$$  
(78.40)

which yields the uncertainties

$$\Delta X_1 = e^{-\epsilon}/\sqrt{2},$$  
(78.41)

$$\Delta X_2 = e^\epsilon/\sqrt{2}.$$  
(78.42)

![Fig. 78.1 The $Q$ parameter for a Schrödinger cat state (78.31) with amplitude $\alpha = 4$](image_url)
In particular, for $\phi = 0$ the squeezed state $|\alpha, r\rangle$ is a minimum uncertainty state for the quadratures $x$ and $p$ with $\Delta x = e^{-r}/\sqrt{2}$ and $\Delta p = e^{r}/\sqrt{2}$. The degree of squeezing in the quadrature $x$ is determined by the squeeze factor $r$.

The average photon number

$$\langle N \rangle = |\alpha|^2 + \sinh^2 r \quad (78.43)$$

of a squeezed state and its photon number variance

$$\langle \Delta N \rangle^2 = |\alpha \cosh r - \alpha^* e^{-2i\phi} \sinh r|^2$$
$$+ 2 \cosh^2 r \sinh^2 r \quad (78.44)$$

contain the coherent contribution $\alpha$ as well as squeezing contributions expressed by $r$ and $\phi$. In particular, for $\phi = 0$, the $Q$ parameter becomes negative for a large enough amplitude $\alpha$ and $r > 0$. The photon number distribution $W_n = <|\alpha, r| n\rangle^2$ becomes narrower than the one for the corresponding coherent state with the same $\alpha$. This sub-Poissonian behavior is one of the nonclassical features of a squeezed state. Furthermore, $W_n$ shows oscillations [78.7] for larger squeezing. The two regimes with sub-Poissonian and oscillating photon statistics $W_n$ are shown in Fig. 78.2.

A second representation of squeezed states has been introduced by Yuen [78.8]. In his notation, a squeezed state is an eigenstate of the operator

$$b = \mu a + v a^\dagger \quad (78.45)$$

with $|\mu|^2 - |v|^2 = 1$ and eigenvalue $\beta$. This eigenstate can be written in the form

$$|\epsilon, \beta\rangle = S(\epsilon) D(\beta)|0\rangle \quad (78.46)$$

which connects the squeezing operator $S(r e^{-2i\phi})$ with the parameters $\mu = \cosh r$ and $v = e^{-2i\phi} \sinh r$. In contrast to the definition (78.35), the displacement operator $D(\beta)$ and the squeezing operator $S(\epsilon)$ are applied now in reversed order. Nevertheless, the two equations (78.35) and (78.46) define the same state if the relation $\alpha = \beta \mu + \beta^* v$ is fulfilled.

Several experiments have demonstrated the generation of squeezed light. Slusher et al. [78.9] obtained squeezing in the side modes of a four-wave mixing process. An optical parametric oscillator below threshold has been used by Wu et al. [78.10] in order to generate squeezed light. Nonclassical features can also be found in a down conversion process. This second-order process creates so-called signal and idler photons from one pump photon. Signal and idler beam are distinguished by frequency or polarization. Heidmann et al. [78.11] have shown that the difference intensity of these twin beams may exhibit reduced quantum fluctuations. Pulsed twin beams also contain reduced noise in the difference of their intensities.

### 78.2.4 Phase States

The problem of a correct quantum mechanical description of phase has a long history in quantum mechanics [78.12]. First attempts to define a quantum phase are due to London and Dirac. The London phase state is

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle \quad (78.47)$$

which is an eigenstate of the exponential phase operator

$$\widehat{e}^{i\phi} = \sum_{n=0}^{\infty} |n\rangle \langle n+1| \quad (78.48)$$

Since this operator is not unitary, it does not define a Hermitian operator $\phi$ for the phase. Nevertheless, many treatments of the phase of a quantum state $|\psi\rangle = \sum c_n |n\rangle$ are based on the London phase distribution

$$P_r(\phi) = |\langle \phi | \psi \rangle|^2 = \left( \frac{1}{2\pi} \right) \sum_{n} |c_n e^{-in\phi}|^2 \quad (78.49)$$

**Fig. 78.2** The photon number distribution $W_n$ of a squeezed state $|\alpha, r\rangle$ with the coherent amplitude $\alpha = 7$. For a squeezing parameter $r = 0$, the Poisson distribution of a coherent state $|\alpha\rangle$ is just visible. When $r$ increases the photon distribution first becomes sub-Poissonian and then oscillatory.
Later treatments [78.13] rely on the Hermitian operators
\[
\sin \phi = \frac{1}{2i} (e^{i\phi} - e^{-i\phi}), \quad (78.50)
\]
\[
\cos \phi = \frac{1}{2} (e^{i\phi} + e^{-i\phi}), \quad (78.51)
\]
for the sine and cosine function of the phase.

### 78.3 Quantum Coherence Theory

This section introduces the correlation functions of the electromagnetic field. Ideal photon correlation measurements can bring out the phenomenon of photon bunching and antibunching.

#### 78.3.1 Correlation Functions

Correlation functions were originally introduced to describe an ideal photodetection process. Glauber [78.16] has presented a treatment based on an absorption mechanism in the detector which is sensitive to the positive frequency part \( E^+ \) of the electric field evaluated at the detector’s space-time position \( x \equiv (x, t) \). This leads to an average field intensity
\[
I(x) = Tr \left[ \rho E^-(x) E^+(x) \right] \quad (78.52)
\]
at point \( x \). Here the density operator \( \rho \) describes the state of the field. The ordering of the operators, i.e., \( E^- E^+ \sim a^\dagger a \), is known as normal ordering with all annihilation operators to the right of all creation operators.

The expression (78.52) now immediately generalizes to the correlation function of first order
\[
G^{(1)}(x_1, x_2) = Tr \left[ \rho E^-(x_1) E^+(x_2) \right], \quad (78.53)
\]
with \( x_1 = (x_1, t_1) \) and \( x_2 = (x_2, t_2) \). The classical interference experiments, such as Young’s double slit experiment, can be described in terms of \( G^{(1)} \). Furthermore, the correlation function of first order is connected to the power spectrum \( S(\omega) \) of a quantized field via the Wiener–Khintchine theorem [78.17]. Under the assumption of a stationary process, i.e., when the autocorrelation function \( \langle E^-(t) E^+(t') \rangle \) depends only on the time difference \( t = t' \), then
\[
S(\omega) = \frac{1}{2\pi} \int_0^\infty \text{d}t \langle E^-(t) E^+(0) \rangle e^{-i\omega t} + c.c. \quad (78.54)
\]
This relation between the spectrum and the first-order correlation function is known as Wiener–Khintchine theorem.

Recently [78.14] a Hermitian phase operator was constructed starting from the phase state (78.47), restricted to a finite Hilbert space. An operational phase description has been proposed [78.15] in which a classical phase measurement is translated to the quantum realm by using an eight-port homodyne detector.

In order to analyze the Hanbury–Brown and Twiss experiment [78.18] it is necessary to define higher order correlation functions. The general \( n \)th order correlation function is defined by
\[
G^{(n)}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) = Tr \left[ \rho E^-(x_1) \cdots E^-(x_n) \times E^+(x_{n+1}) \cdots E^+(x_{2n}) \right], \quad (78.55)
\]
where the field operators are again normal ordered. These correlation functions fulfill a generalized Schwartz inequality
\[
G^{(1)}(x_1, x_2) G^{(1)}(x_2, x_1) \geq \left| G^{(1)}(x_1, x_2) \right|^2, \quad (78.56)
\]
which becomes, for the \( n \)th order functions,
\[
G^{(n)}(x_1, \ldots, x_n, x_n, \ldots, x_1) \times G^{(n)}(x_{n+1}, \ldots, x_{2n}, x_{2n}, \ldots, x_{n+1}) \geq \left| G^{(n)}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \right|^2. \quad (78.57)
\]
A field is said to be first-order coherent when its normalized correlation function
\[
g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\left| G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2) \right|^{1/2}} \quad (78.58)
\]
satisfies \( g^{(1)}(x_1, x_2) = 1 \). In a Young type experiment, this case gives maximum fringe visibility. A more general definition of first order coherence is the condition that \( G^{(1)}(x_1, x_2) \) factorizes
\[
G^{(1)}(x_1, x_2) = \hat{g}^*(x_1) \hat{g}(x_2), \quad (78.59)
\]
where \( \hat{g} \) denotes some complex function. This definition can be readily generalized to the \( n \)th order case. The \( n \)th order coherence applies when the relation
\[
G^{(n)}(x_1, \ldots, x_{2n}) = \hat{g}^*(x_1) \cdots \hat{g}^*(x_n) \hat{g}(x_{n+1}) \cdots \hat{g}(x_{2n}) \quad (78.60)
\]
holds. A field in a coherent state possesses \( n \)th order coherence.

### 78.3.2 Photon Correlations

The Young experiment demonstrates the appearance of first-order correlations. However, experiments that can distinguish between the classical and quantum domains have to be based on measurements of second-order correlations. These experiments are of the Hanbury-Brown and Twiss type, and determine the second-order correlations. These experiments are of great interest in experiments that can distinguish between the classical and quantum domains; see for example (78.195).

If \( g^{(2)}(0) > 1 \), the photons show a tendency to arrive in bunches, an effect known as photon bunching. This effect has been observed for chaotic light. The opposite situation with \( 0 \leq g^{(2)}(0) < 1 \) demonstrates the reverse effect, namely photon antibunching. As seen from (78.63), this is a regime only accessible to non-classical light. An example is given by the resonance fluorescence of a two-level atom, treated in Sect. 78.13.

### 78.4 Photodetection Theory

So far we have used a very simple theory of photodetection: any absorbed photon leads to a photoelectric emission which can be observed. But in any real experiment, these photons are counted over some time interval \( T \) and the observed photoelectric emissions are dominated by two statistics: (i) the statistics of photoelectric emission which is also present for a classical field and (ii) the specific quantum statistics of a quantized field. A detailed discussion of the quantum theory of photoelectric detection has been given by Kelley and Kleinert [78.19]. A central result is the formula

\[
p(n, t, T) = \left\{ \frac{I}{n!} \exp(-I) \right\}
\]

for the probability of counting \( n \) photoelectrons in the time interval from \( t \) to \( t+T \). This photocounting distribution contains the integrated intensity operator

\[
I = \eta \int_{t}^{t+T} \mathrm{d}t' \ E^{-}(t') \ E^{+}(t')
\]

containing the quantum efficiency \( \eta \) of the detector. The notation \( \left\{ \cdots \right\} \) indicates a quantum average where the operators have to be normally ordered and time ordered. This operator ordering reflects the process on which a photodetector is based. It annihilates or absorbs photons, one after the other. A good treatment of photoelectric detection can be found in [78.20].

#### 78.4.1 Homodyne and Heterodyne Detection

These detection methods allow the extraction of specific quantum features of a single mode quantum field,
the signal field. Figure 78.3 summarizes the principle of optical homodyning. Two quantum fields described by the annihilation operators $a$ and $b$ are mixed at a 50/50 beam splitter $BS$. Both fields have the same frequency. The mode $a$ represents the signal mode whose quantum state is given by the density operator $\rho$. Mode $b$ serves as a reference field, the local oscillator. The coherent state $|\alpha\rangle = |\alpha| e^{i\theta}$ determines the quantum state of the local oscillator. Two ideal photodetectors 1 and 2 measure the number of photons in the output modes of the beam splitter. For a highly excited coherent state, i.e., a classical local oscillator, the statistics of the photocurrent $\Delta I$ can be described by the moments of the signal mode operator

$$X_\theta = \frac{1}{\sqrt{2}} (a e^{-i\theta} + a^\dagger e^{i\theta}) .$$  (78.68)

For example, the photocurrent difference

$$\langle \Delta I \rangle \sim \langle X_\theta \rangle = \text{Tr}(\rho X_\theta)$$  (78.69)

is proportional to the expectation value of $X_\theta$. In particular, for $\theta = 0$ and $\theta = \frac{\pi}{2}$ one is able to measure all the moments of the two quadratures $s$ and $p$ (78.13) and (78.14) of the signal mode. In general, the statistics of the photocurrent $\Delta I$ reveal the probability distribution

$$P_r(X_\theta) = \langle X_\theta | \rho | X_\theta \rangle$$  (78.70)

of the observable $X_\theta$ when the signal mode is in the state $\rho$. The states

$$|X_\theta\rangle = \pi^{-1/4} \exp \left[-X^2_\theta/2\right]$$

$$\times \sum_{n=0}^\infty \frac{1}{\sqrt{2^n n!}} H_n(X_\theta) e^{i\theta}|n\rangle$$  (78.71)

are eigenstates of the operator $X_\theta$, and are known as rotated quadrature states.

The heterodyne technique [78.21,22] relies on a similar mixing of a signal field with a local oscillator at a beam splitter, but this time the local oscillator frequency is offset by the intermediate frequency $\Delta \omega$ with respect to the frequency $\omega_0$ of the signal mode. Filters select the beat frequency components in the photocurrent of the detectors. This photocurrent contains the quantum statistics of the two quadratures of the signal field [78.21,22].

### 78.5 Quasi-Probability Distributions

Quasi-probability distributions play an important role in quantum optics for three reasons. First, they are a complete representation of the density operator of a quantum field. Second, they allow one to calculate expectation values in the spirit of classical statistical physics. Third, they offer the possibility of converting a master equation for the density operator into an equivalent $c$-number partial differential equation. In this section, we relate a specific quasi-probability function to a specific operator ordering.

#### 78.5.1 $s$-Ordered Operators

A normally ordered product of $a$ and $a^\dagger$ is a product of the form $(a^\dagger)^n a^m$: the annihilation operators $a$ stand to the right of the creation operators $a^\dagger$. In an antinormally ordered product like $a^n (a^\dagger)^m$, the order of $a$ and $a^\dagger$ has changed. A generalized $s$-ordered product can be defined as

$$\left\{a^n (a^\dagger)^m\right\}_s \equiv \left(\frac{\partial}{\partial \xi^*}\right)^{m} \left(\frac{-\partial}{\partial \xi}\right)^{n}$$

$$\times D(\xi, \xi^*, s) \bigg|_{\xi=\xi^*=0}$$  (78.72)

with the generalized displacement operator

$$D(\xi, \xi^*, s) \equiv \exp \left(\xi a^\dagger - \xi^* a + \frac{1}{2} s \xi \xi^*\right) .$$  (78.73)

For $s = 1$ we find again normal ordering. The values $s = 0$ and $s = -1$ produce symmetric and antinormal ordered products. As an example we note $[a^\dagger a]_s = a^\dagger a - (s-1)/2$. 

**Fig. 78.3** The principle of optical homodyning
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78.5.3 The Wigner Function

This quasi-probability was first introduced by Wigner [78.29] and may be defined as the distribution function for a symmetrically ordered operator product which is obtained in the case \( s = 0 \). The Wigner function plays an important role in other branches of physics, such as quantum chaology, and in particular in any semiclassical phenomenon when one considers the transition from quantum mechanics to classical mechanics.

Consider the Wigner function of a quantum mechanical particle of position

\[
x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) ,
\]

and momentum

\[
p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) .
\]

The Wigner function may be written in terms of position and momentum variables

\[
W(x, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy \ e^{-2ipy/\hbar} \langle x + y | \rho | x - y \rangle ,
\]

where \( |x \pm y\rangle \) denotes position eigenstates. The \( s = 0 \) Cahill–Glauber definition and the above definition of the Wigner function are related by

\[
W(x, p) = \frac{1}{2} W_\alpha (x^s + ip) / \sqrt{2m\hbar\omega}, \ s = 0 .
\]
The position and momentum distributions of a particle, or equivalently the quadrature distributions in the case of a quantized field mode, are

\[ Pr(x) = \int_{-\infty}^{\infty} W(x, p) \, dp, \quad (78.88) \]

\[ Pr(p) = \int_{-\infty}^{\infty} W(x, p) \, dx. \quad (78.89) \]

Furthermore, the scalar product

\[ |\langle \psi_1 | \psi_2 \rangle|^2 = 2\pi \hbar \int \int dx \, dp \, W_{\psi_1}(x, p) \times W_{\psi_2}(x, p) \]

(78.90)

of two quantum states is expressed by the phase space overlap of the two corresponding Wigner functions. Consequently, any Wigner function \( W(x, p) \) has to obey the necessary condition

\[ \int \int dx \, dp \, W(x, p) \times W_{\psi_1}(x, p) \geq 0 \]

(78.91)

for all \( W_{\psi_1} \) representing a pure state. For a normalized state \( |\psi_i\rangle \),

\[ \int \int dx \, dp \, |W_{\psi_i}(x, p)|^2 = \frac{1}{2\pi \hbar}. \]

(78.92)

Instead of solving the Schrödinger equation for the dynamics of a massive particle in a potential \( V(x) \), we can try to solve the equation

\[ \frac{\partial W}{\partial \tau} = -\frac{p}{m} \frac{\partial W}{\partial x} + \sum_{r=1,3,5,7,\ldots} \frac{1}{r!} \left( \frac{i\hbar}{2} \right)^{r-1} \sqrt{2} \frac{\partial^r V}{\partial x^r} \frac{\partial W}{\partial p^r} \]

(78.93)

for its Wigner function \( W(x, p, \tau) \). Note that here only the odd derivatives of the potential \( V \) enter. This equation is the quantum analogue of the classical Liouville equation, to which it reduces in the limit of \( \hbar \rightarrow 0 \). However, the initial distribution \( W(x, p, \tau = 0) \) has to be a Wigner function in the sense of (78.86).

Furthermore the Wigner function of an energy eigenfunction in the potential \( V(x) \) may be obtained from the equations

\[ \frac{p^2}{2m} + V(x) = \frac{\hbar^2}{8m} \frac{\partial^2 \psi^2}{\partial x^2} - \frac{\hbar^2}{8} \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial p^2} \]

\[ + \int_{-\infty}^{\infty} \left( \frac{i\hbar}{2} \right)^{r-1} \frac{\partial^r V}{\partial x^r} \frac{\partial W}{\partial p^r} = EW(x, p). \]

(78.94)

and

\[ \left( -\frac{p}{m} \frac{\partial}{\partial x} + \frac{\partial V}{\partial x} \frac{\partial}{\partial p} \right) W(x, p) \]

\[ + \sum_{r=3,5,7,\ldots} \frac{1}{r!} \left( \frac{i\hbar}{2} \right)^{r-1} \frac{\partial^r V}{\partial x^r} \frac{\partial W}{\partial p^r} = 0. \]

(78.95)

The Wigner function has negative parts for most quantum states. For example, the Wigner function of a Fock state \( |n\rangle \),

\[ W_{\langle n \mid}(\vec{x}, \vec{p}) = \frac{(1-n)!}{\pi} e^{-\vec{x}^2/2} e^{-\vec{p}^2} L_n(2\vec{x}^2 + 2\vec{p}^2), \]

(78.96)

clearly becomes negative due to the oscillating Laguerre polynomial \( L_n \) as shown in Fig. 78.4. Note that we have introduced the dimensionless position \( \vec{x} = \sqrt{\mu\hbar/\omega} x \) and momentum \( \vec{p} = 1/\sqrt{\mu m\hbar} p \).

On the other hand, the Wigner function

\[ W_{\langle \alpha \mid}(\vec{x}, \vec{p}) = \frac{1}{\pi} \exp \left\{ -2^\alpha \left[ \vec{x} - \sqrt{2} \text{Re}(\alpha) \right]^2 - e^{-2\alpha} \left[ \vec{p} - \sqrt{2} \text{Im}(\alpha) \right]^2 \right\} \]

(78.97)

of a squeezed state (78.35) is always positive as shown in Fig. 78.5. It is a long thin ellipse in phase space (i.e. a Gaussian cigar). Concerning the negative parts of the Wigner function, the Hudson theorem (78.30) states that a necessary and sufficient condition for the Wigner function of a pure state \( |\psi\rangle \) to be nonnegative is that it can be described by a wave function of the
78.5.4 The $Q$ Function

The $Q$ function is defined by the diagonal matrix elements

$$Q(\alpha) = \langle \alpha | \rho | \alpha \rangle / \pi$$  \hspace{1cm} (78.101)

of the density operator $\rho$, where $|\alpha\rangle$ denotes a coherent state. The $Q(\alpha)$ function is always a positive and bounded function, which exists for any density operator $\rho$. The $Q$ function is also known as Husimi’s function. It allows one to calculate expectation values of antinormally ordered operator products of the form

$$\langle a^n (a^\dagger)^m \rangle = \int d^2 \alpha \alpha^n (\alpha^*)^m Q(\alpha).$$  \hspace{1cm} (78.102)

Moreover, since the $Q$ function corresponds to the case $s = -1$ of the Cahill–Glauber distribution,

$$Q(\alpha) = W(\alpha, -1).$$  \hspace{1cm} (78.103)

78.5.5 Relations Between Quasi-Probabilities

In general, the relation

$$W(\alpha, s) = \frac{2}{\pi (s' - s)} \int d^2 \beta \exp \left(-\frac{2|\alpha - \beta|^2}{s' - s} \right) \times W(\beta, s')$$  \hspace{1cm} (78.104)

holds between two Cahill–Glauber distributions with the parameters $s' > s$. In particular, the non-negative $Q$ function

$$Q(\alpha) = \frac{2}{\pi} \int d^2 \beta \exp[-2|\alpha - \beta|^2] W(\beta, s = 0)$$  \hspace{1cm} (78.105)

turns out to be a smoothed Wigner function $W(\beta, s = 0)$. It is this smoothing process that washes out possible negative parts in the Wigner function.

78.6 Reservoir Theory

Reservoir theory treats the interaction of one system with a few degrees of freedom, called the system, with another system with many degrees of freedom, called the reservoir. A typical application of reservoir theory is a microscopic theory of damping: the system interacts with a reservoir, called the heat bath. The system dissipates energy into the heat bath whereas the heat bath introduces additional fluctuations to the system. Since the present chapter focuses on quantized field effects, the reservoir consists of the many modes of the radiation field in free space. Such a reservoir is modeled by a large number of independent harmonic...
oscillators
\[ H_r = \sum_i \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right) , \]  
(78.106)

where \( b_i \) and \( b_i^\dagger \) are the annihilation and creation operators for the \( i \)th harmonic oscillator of the reservoir. For convenience the interaction with the system is frequently approximated by a Hamiltonian of the form
\[ H_{\text{int}} = \hbar \sum_i \left( g_i A b_i^\dagger + g_i^* A^\dagger b_i \right) , \]  
(78.107)

where \( A \) is an operator of the small system and \( g_i \) is the coupling strength of this system to the \( i \)th oscillator of the reservoir. For example, \( A \) may be an annihilation operator if the system is a harmonic oscillator or a Pauli spin matrix in the case of a two-level atom coupled to the free space radiation field.

Reservoir theory has important applications, and a detailed discussion can be found in various books, for example [78.17, 20, 27, 28, 31–33].

78.6.1 Thermal Reservoir
The most commonly used reservoir is the thermal reservoir or thermal heat bath. Its characteristic properties are
\[ \langle b_i \rangle = \langle b_i^\dagger \rangle = \langle b_i^\dagger b_i \rangle = \langle b_i b_i^\dagger \rangle = 0 , \]  
(78.108)
\[ \langle b_i^\dagger b_j \rangle = \pi_i \delta_{ij} . \]  
(78.109)

Here
\[ \pi_i = \frac{1}{\exp(\hbar \omega_i / k_B T) - 1} \]  
(78.110)
is the average number of photons at frequency \( \omega_i \), \( T \) is the temperature of the reservoir, and \( k_B \) denotes the Boltzmann constant.

78.6.2 Squeezed Reservoir
Another example of a reservoir is a squeezed vacuum or squeezed reservoir. If, for example, multiwave mixing is used to squeeze the radiation field, conjugate pairs of the reservoir operators \( b_i \) are correlated. Therefore, the expectation values \( \langle b_i^\dagger b_i \rangle \) and \( \langle b_i b_i^\dagger \rangle \) may be nonvanishing. Apart from the average number \( \pi_i \) of photons at frequency \( \omega_i \), which take into account nonvanishing expectation values \( \langle b_i^\dagger b_i \rangle \), additional complex squeezing parameters are needed to describe the reservoir [78.28, 33, 34]. The characterization of a squeezed reservoir based on noise operators is discussed in Sect. 78.10.

78.7 Master Equation
In quantum mechanics, density operators are used to describe mixed states, and are discussed in Chapt. 7. Here we introduce the concept of the reduced density operator
\[ \rho_s = \text{Tr}_r(\rho_{sr}) , \]  
(78.111)

which is the density operator \( \rho_{sr} \) of the complete system traced over the degrees of freedom of the reservoir. The equation of motion for \( \rho_s \) in the Schrödinger picture is
\[ \dot{\rho}_s(t) = -\frac{i}{\hbar} \text{Tr}_r \left\{ [H_{sr}, \rho_{sr}(t)] \right\} . \]  
(78.112)

In the Born–Markov approximation the trace over the reservoir can be evaluated and leads to an equation of motion for \( \rho_s \) which no longer contains reservoir operators. This equation of motion is usually called the master equation. The Born–Markov approximation consists of two different parts:

1. Born approximation: The coupling to the reservoir is assumed to be sufficiently weak to allow a perturbative treatment of the interaction between the reservoir and the system.

2. Markov approximation: The correlations of the reservoir are assumed to decay very rapidly on a typical time scale of the system, or equivalently, the reservoir has a very broad spectrum. This approximation involves the assumption that the modes of the reservoir are spaced closely together, so that the frequency \( \omega_i \) is a smooth function of \( i \).

Since a general treatment is rather technical, we consider two typical examples. A more general discussion can be found in [78.17, 20, 27, 28, 31–33].
78.7.1 Damped Harmonic Oscillator

The universally accepted Hamiltonian in nonrelativistic QED for a harmonic oscillator of frequency \( \omega \) coupled to a reservoir consisting of a large number of harmonic oscillators is given by the total Hamiltonian\(^{[78.35, 36]}\)

\[
H_a = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + \sum_i \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right) + H_k + H_{\alpha} , \quad \text{(78.113)}
\]

with the linear coupling term

\[
H_k = \hbar \sum_i g_i \left( a + a^\dagger \right) \left( b_i + b_i^\dagger \right) , \quad \text{(78.114)}
\]

and the self-interaction term

\[
H_{\alpha} = \sum_i \frac{\hbar^2 g_i^2}{\omega_i} \left( a + a^\dagger \right)^2 . \quad \text{(78.115)}
\]

The approach used in quantum optics is to drop the term \( H_{\alpha} \) and to make the rotating-wave approximation, that is, to drop the terms \( a b_i \) and \( a^\dagger b_i^\dagger \), see also Chapt. 68. Then the approximate total Hamiltonian reads

\[
H_a = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) + \sum_i \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right) + \hbar \sum_i g_i \left( a b_i^\dagger + a^\dagger b_i \right) . \quad \text{(78.116)}
\]

Despite the problems with this approximate Hamiltonian (see Sect. V.D of \([78.35, 36]\) for a discussion) we adopt it in the present context because it leads to the widely used master equation for the damped harmonic oscillator. We consider two reservoirs: a thermal bath and a squeezed bath.

**Harmonic Oscillator in a Thermal Bath**

Within the Born–Markov approximation the master equation is

\[
\dot{\rho} = \frac{1}{2} \gamma (\bar{n} + 1) \left( 2a^\dagger a^\dagger a - a^\dagger a - a a^\dagger \right) + \frac{1}{2} \gamma \bar{n} \left( 2a^\dagger a a - a a^\dagger a - a^\dagger a a^\dagger \right) + \frac{1}{2} \gamma 2 \bar{n} a^\dagger a - a a^\dagger a^2 , \quad \text{(78.117)}
\]

where

\[
\rho(t) = e^{i \omega t (\hat{a}^\dagger - \hat{a})} \rho_0 (t) e^{-i \omega t (\hat{a}^\dagger - \hat{a})} \quad \text{(78.121)}
\]

is the reduced density operator in the interaction picture. The damping constant \( \gamma \) is given by

\[
\gamma = 2 \pi \mathcal{D} (\omega) |g(\omega)|^2 . \quad \text{(78.119)}
\]

where \( g(\omega) \) denotes the coupling strength at frequency \( \omega \). The number of thermal photons at frequency \( \omega \) is

\[
\bar{n} = \frac{1}{\exp (\hbar \omega / k_B T) - 1} . \quad \text{(78.120)}
\]

Thus the Born–Markov approximation replaces the discrete reservoir modes by a continuum of modes with a density \( \mathcal{D} (\omega) \).

**Harmonic Oscillator in a Squeezed Bath**

Within the Born–Markov approximation, the reduced density operator \( (78.118) \) in the interaction picture satisfies the master equation

\[
\dot{\rho} = \frac{1}{2} \gamma (\bar{m} + 1) \left( 2a^\dagger a^\dagger a - a^\dagger a a - a^\dagger a^\dagger \right) + \frac{1}{2} \gamma \bar{m} \left( 2a^\dagger a a a - a a^\dagger a^\dagger a - a^\dagger a a^\dagger \right) + \frac{1}{2} \gamma 2 \bar{m} a^\dagger a - a^\dagger a a^2 , \quad \text{(78.121)}
\]

Here \( \gamma \) is again given by (78.119). The squeezed reservoir is characterized by a real number \( \bar{n} \) and a complex number \( \bar{m} \). Physically \( \bar{n} \) is the number of photons at frequency \( \omega \), i.e., similar to the thermal reservoir, it measures the average energy at frequency \( \omega \). The complex number \( \bar{m} \) determines the amount of squeezing. In general, the positivity of the density operator requires

\[
|\bar{m}|^2 \leq \bar{n} (\bar{n} + 1) . \quad \text{(78.122)}
\]

A more quantitatively definition of \( \bar{n} \) and \( \bar{m} \) in terms of noise operators is given in Sect. 78.10.

78.7.2 Damped Two-Level Atom

The interaction of a two-level atom with a classical electromagnetic field is already discussed in Chapt. 68. For a quantum mechanical treatment of the field we only have to replace the classical field by its quantum mechanical counterpart (78.9). We then find in the rotating-wave approximation (Chapt. 68), that the dynamics of a two-level atom with a transition frequency \( \omega_0 \) coupled to a reservoir consisting of a large number of harmonic oscillators is approximately described by the total Hamiltonian

\[
H_a = \frac{\hbar}{2} \omega_0 \sigma_z + \sum_i \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right) + \hbar \sum_i \left( g_i \sigma_+ b_i^\dagger + g_i^* \sigma_- b_i \right) . \quad \text{(78.123)}
\]
where

\[ \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (78.124) \]

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (78.125) \]

Again, two reservoirs are considered: a thermal bath and a squeezed bath.

**Two-Level Atom in a Thermal Bath**

Within the Born–Markov approximation, the master equation is

\[
\dot{\rho} = \frac{1}{2} \gamma (\vec{\pi} + 1) (2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_-) + \frac{1}{2} \gamma \vec{\pi} (2\sigma_+ \rho \sigma_- - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+) \quad (78.126)
\]

where \( \gamma, \vec{\pi} \) and \( \vec{\pi} \) have the same meaning as in (78.121).

### 78.8 Solution of the Master Equation

#### 78.8.1 Damped Harmonic Oscillator

We consider only a thermal reservoir and present the solution of the master equation (78.117). For \( \vec{\pi} = 0 \) it can be solved in terms of coherent states, see (78.26). For \( \vec{\pi} \neq 0 \) we give solutions in terms of quasi-probability distributions.

**Coherent States**

For \( \vec{\pi} = 0 \), which is a good approximation for optical frequencies, if the system is initially in a coherent state \( |\alpha_0\rangle \) with a density operator

\[ \rho(t_0) = |\alpha_0\rangle \langle \alpha_0 |, \quad (78.129) \]

then there exists a simple analytical solution of the master equation (78.117)

\[ \rho(t) = |\alpha_0 e^{-\gamma(t-t_0)/2} \rangle \langle \alpha_0 e^{-\gamma(t-t_0)/2} |, \quad (78.130) \]

A coherent state thus remains a coherent state with an exponentially decaying amplitude \( \alpha_0 e^{-\gamma(t-t_0)/2} \). According to (78.78) a general solution

\[ \rho(t) = \int d^2 \alpha_0 P(\alpha_0) |\alpha_0 e^{-\gamma(t-t_0)/2} \rangle \langle \alpha_0 e^{-\gamma(t-t_0)/2} |, \quad (78.131) \]

can be constructed for an initial density operator

\[ \rho(t_0) = \int d^2 \alpha_0 P(\alpha_0) |\alpha_0\rangle \langle \alpha_0 |. \quad (78.132) \]

If the system is initially in a superposition

\[ |\psi(t_0)\rangle = \sum_i c_i |\alpha_i\rangle \quad (78.133) \]

of coherent states, the time evolution is given by

\[
\rho(t) = \sum_{i,k} c_i c_k^* \exp \left[ -\frac{1}{2} \left( 1 - e^{-\gamma(t-t_0)} \right) |\alpha_i - \alpha_k|^2 \right] \times \exp \left[ i \left( 1 - e^{-\gamma(t-t_0)} \right) \text{Im} (\alpha_i \alpha_k^*) \right] \times |\alpha_i e^{-\gamma(t-t_0)/2} \rangle \langle \alpha_k e^{-\gamma(t-t_0)/2} |. \quad (78.134)
\]

For \( \gamma(t-t_0) \ll 1 \), the interference terms \( |\alpha_i\rangle \langle \alpha_k |, i \neq k \) decay with an effective decay constant \( \gamma |\alpha_i - \alpha_k|^2 / 2 \). This is the damping constant is modified by the separation of the two coherent states in phase space.

**Fokker–Planck Equation**

A widely used procedure for solving the master equation for a damped harmonic oscillator, (78.117), or for similar problems, is to derive an equation of motion for the quasi-probability distributions \( W(\alpha, \alpha^*; t) \) defined in (78.76) from the master equation. The operators \( a \) and \( a^\dagger \) are replaced by appropriate differential operators. The substitution rules can be derived
from (78.73), (78.74) and (78.76) and are

\[
\rho \left( a^\dagger \right)^k a^\ell \rho \rightarrow \left( a^\ast - \frac{s+1}{2} \frac{\partial}{\partial a^\ast} \right)^k \left( a - \frac{s-1}{2} \frac{\partial}{\partial a} \right) \ell W, \\
\rho \left( a^\dagger \right)^k a^\ell \rho \rightarrow \left( a^\ast - \frac{s-1}{2} \frac{\partial}{\partial a^\ast} \right)^k W, \\
a^\dagger a \rho \rightarrow \left( a^\ast - \frac{s+1}{2} \frac{\partial}{\partial a^\ast} \right) W, \\
a^\dagger a \rho \rightarrow \left( a^\ast - \frac{s-1}{2} \frac{\partial}{\partial a^\ast} \right) W, \\
a^\ell a \rho \rightarrow \left( a - \frac{s+1}{2} \frac{\partial}{\partial a} \right)^\ell W, \\
a^\ell a \rho \rightarrow \left( a - \frac{s-1}{2} \frac{\partial}{\partial a} \right)^\ell W. \\
\rho \rightarrow \left( a^\ast - \frac{s-1}{2} \frac{\partial}{\partial a^\ast} \right) \left( a - \frac{s+1}{2} \frac{\partial}{\partial a} \right) W. \\
(78.135)
\]

The time-dependent solution of this Fokker–Planck equation has the form

\[
W(\alpha, \alpha^\ast, t; s) = \int G(\alpha, \alpha^\ast, t|\alpha', \alpha'^\ast, t'; s) \times W(\alpha', \alpha'^\ast, t'; s) \, d^2 \alpha', \\
(78.138)
\]

where

\[
G(\alpha, \alpha^\ast, t|\alpha', \alpha'^\ast, t'; s) = \exp \left( -\frac{\left| \alpha - \alpha' \right|^2}{\pi \tilde{\Pi}_s \left( 1 - e^{-2\gamma(t-t')} \right)} \right) \\
(78.139)
\]

is the Green’s function of the Fokker–Planck equation (78.136). The steady-state solution is

\[
W(\alpha, \alpha^\ast, t \rightarrow \infty; s) = \frac{1}{\pi \tilde{\Pi}_s} e^{-|\alpha|^2/\tilde{\Pi}_s}, \\
(78.140)
\]

which is the distribution function of a harmonic oscillator in thermal equilibrium with a reservoir of temperature \( T \).

### 78.8.2 Damped Two-Level Atom

The density operator

\[
\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \\
(78.141)
\]

for a two-level atom can be written as

\[
\rho = \left( \frac{1}{2} \right) \begin{pmatrix} 1 + \langle \sigma_z \rangle & \langle \sigma_z \rangle \\ \langle \sigma_z \rangle & 1 - \langle \sigma_z \rangle \end{pmatrix}. \\
(78.142)
\]

Thus, a two-level atom is completely described by the expectation values

\[
\langle \sigma_z \rangle = \rho_{ee} - \rho_{gg}, \\
\langle \sigma_+ \rangle = \rho_{ge}, \\
\langle \sigma_- \rangle = \rho_{eg}. \\
(78.143)
\]

Hence the master equation (78.128) can be cast into the equations of motions for these expectation values

\[
\frac{d}{dt} \langle \sigma_+ \rangle = -\gamma \left( \langle \sigma_+ \rangle + \frac{1}{2} \right) \langle \sigma_- \rangle - \gamma \pi \langle \sigma_- \rangle, \\
(78.137)
\]

\[
\frac{d}{dt} \langle \sigma_- \rangle = -\gamma \left( \langle \sigma_- \rangle - \frac{1}{2} \right) \langle \sigma_+ \rangle - \gamma \pi \langle \sigma_+ \rangle. \\
(78.137)
\]
\[
\frac{d}{dt} \langle \sigma_- \rangle = -\gamma \left( \frac{\pi}{2} + \frac{1}{2} \right) \langle \sigma_- \rangle - \gamma |\sigma_- \rangle |\sigma_+ \rangle ,
\]
\[
\frac{d}{dt} \langle \sigma_+ \rangle = -2\gamma \left( \frac{\pi}{2} + \frac{1}{2} \right) \langle \sigma_+ \rangle - \gamma ,
\] (78.144)

which can easily be solved for arbitrary initial conditions. In contrast to a thermal reservoir (\(m = 0\)), a squeezed reservoir results in two different transverse decay constants \(\gamma (\frac{\pi}{2} + \frac{1}{2} + |m|)\) and \(\gamma (\frac{\pi}{2} + \frac{1}{2} - |m|)\) [78.34].

### 78.9 Quantum Regression Hypothesis

In the Schrödinger picture, time-dependent expectation values for system operators \(A_j\) can be calculated from the reduced density operator \(\rho_s(t)\) via
\[
\langle A_j \rangle = \text{Tr}_s \left\{ A_j \rho_s(t) \right\} .
\] (78.145)
The reduced density operator, however, is not sufficient to calculate two-time correlation functions such as \(\langle A_j(t + \tau) A_k(t) \rangle\). For a definition of two-time correlation functions, the Heisenberg picture is more appropriate. Here, expectation values follow from
\[
\langle A_j \rangle = \text{Tr}_s \left[ U_{st}(t, t_0) A_j(t_0) U_{st}(t, t_0) \rho_{st}(t_0) \right] ,
\] (78.146)
where \(U_{st}(t, t_0)\) describes the unitary time evolution of the complete system and \(\rho_{st}(t_0)\) is the density operator in the Heisenberg picture. Similarly, two-time correlation functions such as \(\langle A_j(t + \tau) A_k(t) \rangle\) can be defined as
\[
\langle A_j(t + \tau) A_k(t) \rangle = \text{Tr}_s \left[ U_{st}(t, t_0) A_j(t_0) U_{st}(t + \tau, t_0) \right.
\]
\[
\times U_{st}^\dagger(t, t_0) A_k(t_0) U_{st}(t, t_0) \rho_{st}(t_0) \left. \right] .
\] (78.147)
The quantum regression hypothesis avoids the calculation of \(U_{st}(t, t_0)\). Two equivalent formulations exist, one based on the master equation for \(\rho_s\) and another based on the equation of motion for the expectation values \(\langle A_j \rangle\), see for example [78.20, 27, 28, 31, 33].

### 78.9.1 Two-Time Correlation Functions and Master Equation

It follows from their definition (78.147) in the Heisenberg picture that two-time correlation functions \(\langle A_j(t + \tau) A_k(t) \rangle\) for system operators \(A_j\) and \(A_k\) can be calculated with the help of the operator
\[
R_s(t + \tau, t) = \text{Tr}_s \left[ U_{st}(t + \tau, t) \right.
\]
\[
\times A_k \rho_{st}(t) U_{st}^\dagger(t + \tau, t) \left. \right] ,
\] (78.148)
where \(U_{st}(t + \tau, t)\) describes the unitary time evolution of the complete system between \(t\) and \(t + \tau\). We find
\[
\langle A_j(t + \tau) A_k(t) \rangle = \text{Tr}_s \left[ A_j R_s(t + \tau, t) \right] .
\] (78.149)

Note, that in (78.148) and (78.149) we interpret \(A_j\) and \(A_k\) as operators in the Schrödinger picture and have omitted the argument \(t_0\). Because the reduced density operator
\[
\rho_s(t) = \text{Tr}_{\tau} \left[ U_{st}(t, t_0) \rho_{st}(t_0) U_{st}^\dagger(t, t_0) \right] \quad \text{(78.150)}
\]
satisfies the master equation, it is plausible to assume, that when the time derivative is taken with respect to \(\tau\), the operator \(R_s(t + \tau, t)\) also satisfies the master equation for \(\rho_s\) subject to the initial condition \(R_s(t, t) = A_k \rho_s(t)\). However, this requires the additional assumption that the approximations made in the derivation of the master equation for \(\rho_s(t)\) are also valid for \(R_s(t + \tau, t)\).

### 78.9.2 Two-Time Correlation Functions and Expectation Values

A second formulation of the quantum regression hypothesis asserts that two-time correlation functions \(\langle A_j(t + \tau) A_k(t) \rangle\) obey
\[
\frac{\partial}{\partial \tau} \langle A_j(t + \tau) A_k(t) \rangle = \sum_\ell G_{j\ell}(\tau) \langle A_k(t + \tau) A_\ell(t) \rangle ,
\] (78.151)
provided that the expectation values of a set of system operators \(A_j\) satisfy
\[
\frac{\partial}{\partial t} \langle A_j(t) \rangle = \sum_\ell G_{j\ell}(t) \langle A_\ell(t) \rangle .
\] (78.152)
This is the form of the quantum regression hypothesis that was first formulated by Lax [78.38].

The equivalence of the two formulations follows from the interpretation of \(R_s(t + \tau, t)\) on the right side of (78.149) as a “density operator”. Then \(\text{Tr}_s \left\{ A_j R_s(t + \tau, t) \right\}\) is an “expectation value” for which we assume that (78.152) is valid, i.e.,
\[
\frac{\partial}{\partial \tau} \text{Tr}_s \left\{ A_j R_s(t + \tau, t) \right\} = \sum_\ell G_{j\ell}(\tau) \text{Tr}_s \left\{ A_\ell R_s(t + \tau, t) \right\} .
\] (78.153)
According to (78.149), this is identical to (78.151).
Quantum Noise Operators

The master equation is based on the Schrödinger picture in quantum mechanics: the state of the system described by a density operator is time-dependent, whereas operators corresponding to observables are time independent. If we use the Heisenberg picture instead and make similar approximations as in the derivation of the master equation, we arrive at equations of motion for the Heisenberg operators, see for example [78.17, 28, 31, 32]. Due to the interaction with a reservoir these equations have additional noise terms and damping terms.

### 78.10.1 Quantum Langevin Equations

Again consider a damped harmonic oscillator. The equation of motion for the annihilation operator

\[ \dot{a}(t) = e^{i\omega(t-t_0)} a(t) \]  
(78.154)

in the interaction picture follows from the Heisenberg equations for the operators \( a, a^\dagger, b, b^\dagger \) and reads

\[ \frac{d\hat{a}}{dr} = -\sum_i |g_i|^2 \int_{t_0}^t e^{-i(a_i-a_\dagger)(t-t')} \hat{a}(t') \, dt' - i \sum_i g_i^* e^{-i(a_i-a_\dagger)t} b(t_0) . \]

In general, the noise operator

\[ F(t) = -i \sum_i g_i^* e^{-i(a_i-a_\dagger)t} b(t_0) \]

is not delta-correlated, and there are also memory effects in (78.155). The noise operator \( F(t) \) can be used to classify the reservoir: if it is delta-correlated, that is, if the reservoir has a very broad spectrum, one speaks of white noise, see below. If the correlation time is finite so that there are memory effects, one speaks of colored noise.

If the spectrum of the noise is very broad (as in the derivation of the master equation for the reduced density operator), the operator \( \hat{a}(t) \) satisfies the quantum Langevin equation

\[ \frac{d\hat{a}}{dr} = -\frac{\gamma}{2} \hat{a}(t) + F(t) , \]

(78.157)

with a damping term \(-\gamma \hat{a}(t)/2\) and a noise term \( F(t) \).

Note that a simple damping equation such as

\[ \hat{a}(t) = -\frac{\gamma}{2} \hat{a}(t) \]

(78.158)

is unphysical since it does not preserve the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). It is the noise term which saves the commutation relation.

For a thermal reservoir with a sufficiently small correlation time, the standard derivations [78.32] give

\[ \langle F(t) \rangle = \langle F^\dagger(t) \rangle = 0 , \]
\[ \langle F(t) F(t') \rangle = \gamma \mu \delta(t-t') , \]
\[ \langle F^\dagger(t) F(t') \rangle = \gamma \mu^* \delta(t-t') , \]
\[ \langle F(t) F^\dagger(t') \rangle = \gamma \mu^* \delta(t-t') , \]
\[ \langle F(t) F^\dagger(t') \rangle = \gamma (\mu + 1) \delta(t-t') , \]
(78.159)

where the averages are taken over the reservoir. The damping constant \( \gamma \) and the number of thermal photons \( \mu \) are given in (78.119) and (78.120). For more general relations, see [78.35, 36], where it is shown explicitly that correlation functions involving the fluctuation forces do not in fact depend on the oscillator frequency. The condition of a sufficiently small reservoir correlation time requires that \( \tau_c \approx h/(k_B T) \) is small compared with the time scales of the systems. The only time scale in (78.157) is \( \gamma^{-1} \). The relevant condition is therefore \( \tau_c \ll \gamma^{-1} \). For typical applications in quantum optics, \( a \) is the annihilation operator and \( a^\dagger \) is the creation operator of a single-mode cavity field. Here one can have quality factors of the cavity on the order of \( Q = \omega_0/\gamma \approx 10^9 \). In terms of the quality factor, the condition of sufficiently small reservoir correlation time requirements \( h\omega_0/(k_B T) \ll Q \). For optical frequencies (\( \omega_0 \approx 3 \times 10^{15} \) Hz) and \( T \approx 300 \) K one has \( h\omega_0/(k_B T) \approx 75 \). In the microwave regime (\( \omega_0 \approx 30 \) GHz) one can have temperatures as low as \( T \approx 3 \) mK and still have \( h\omega_0/(k_B T) \approx 75 \). Therefore the assumption of delta-correlated noise is a good approximation for typical applications in quantum optics.

Similarly, for a squeezed reservoir one has

\[ \langle F(t) \rangle = \langle F^\dagger(t) \rangle = 0 , \]
\[ \langle F(t) F(t') \rangle = \gamma \mu \delta(t-t') , \]
\[ \langle F^\dagger(t) F(t') \rangle = \gamma \mu^* \delta(t-t') , \]
\[ \langle F(t) F^\dagger(t') \rangle = \gamma \mu^* \delta(t-t') , \]
(78.160)

which gives a quantitative definition of the parameters \( \mu \) and \( \mu^* \) in the master equations (78.121) and (78.128). Again, a detailed discussion in [78.35, 36] shows that correlation functions involving the fluctuation forces do not depend on the oscillator frequency.

The Langevin equation (78.157) is based on the use of the approximate Hamiltonian given in (78.116),
i.e., it is based on the rotating-wave approximation and the neglect of self-interaction terms. The corresponding Langevin equation for \( x = \sqrt{\hbar/(2m_{\text{eo}})} (a + a^\dagger) \) may be calculated and, not unexpectedly, it disagrees with the Abraham–Lorentz equation which Ford and O’Connell [78.39] showed could be derived systematically using the exact Hamiltonian (78.113). In fact, Ford and O’Connell showed that an improved equation for the radiating electron (improved in the sense that it is second-order and is not subject to the analyticity problems and the problems with runaway solutions associated with the Abraham–Lorentz equation) may be obtained by generalizing the Hamiltonian (78.113) to include electron structure. The implications following from these different equations are presently under study.

### 78.10.2 Stochastic Differential Equations

In Sect. 78.10.1 we discussed one of the simplest quantum systems with dissipation, the damped harmonic oscillator. For more complicated systems the noise term can also contain system operators. In such cases there are two different ways to interpret the Langevin equation. In order to give a feeling for the two possible interpretations, we discuss the one dimensional classical Langevin equation

\[
\frac{dx}{dt} = g(x, t) + h(x, t) F(t) \tag{78.161}
\]

for the stochastic variable \( x(t) \) with delta-correlated noise \( \langle F(t)F(t') \rangle = \delta(t - t') \). Due to the singular nature of delta-correlated noise, such a Langevin equation does not exist from a strictly mathematical point of view. A mathematically more rigorous treatment is based on stochastic differential equations [78.27, 28, 37]. The variable \( x(t) \) is said to obey a stochastic differential equation

\[
\frac{dx(t)}{dt} = g(x, t) dt + h(x, t) F(t) dt = g(x, t) dt + h(x, t) dW(t) \tag{78.162}
\]

if, for all times \( t \) and \( t_0 \), \( x(t) \) is given by

\[
x(t) = x(t_0) + \int_{t_0}^{t} g(x(t'), t') \, dt' + \int_{t_0}^{t} h[x(t'), t'] \, dW(t') \tag{78.163}
\]

Here the last term is a Riemann–Stieltjes integral defined by

\[
\int_{t_0}^{t} h[x(t'), t'] \, dW(t') = \lim_{n \to \infty} \sum_{i=0}^{n-1} h[x(\tau_i), \tau_i] \left[ W(\tau_{i+1}) - W(\tau_i) \right],
\]

(78.164)

where \( \tau_i \) is in the interval \( (t_i, t_{i+1}) \).

There are two different approaches to such problems: the Ito approach and the Stratonovich approach. They differ in the definition of stochastic integrals.

In the Stratonovich approach, one evaluates \( h[x(\tau_i), \tau_i] \) at \( \tau_i = (t_i + t_{i+1})/2 \), whereas in the Ito approach one evaluates \( h[x(\tau_i), \tau_i] \) at \( \tau_i = t_i \). This slightly different definition of \( \tau_i \) leads to different results because, as a consequence of the delta-correlated noise term, \( x(t) \) is not a continuous path. However, there is a relation between the solution of a Stratonovich stochastic differential equation and an Ito stochastic differential equation. Suppose \( x(t) \) is a solution of the Stratonovich stochastic differential equation

\[
dx(t) = g(x, t) \, dt + h(x, t) \, dW(t) \tag{78.165}\]

Then \( x(t) \) satisfies the Ito stochastic differential equation

\[
dx(t) = \left[ g(x, t) + \frac{1}{2} h(x, t) \frac{\partial h(x, t)}{\partial x} \right] \, dt + h(x, t) \, dW(t) \tag{78.166}
\]

Instead of dealing with stochastic differential equations, one can derive a Fokker–Planck equation for the conditional probability \( P(x, t|x_0, t_0) \). For the Stratonovich stochastic differential equation (78.165), the Fokker–Planck equation is

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ g(x, t) + \frac{1}{2} h(x, t) \frac{\partial h(x, t)}{\partial x} \right] P + \frac{1}{2} \frac{\partial^2}{\partial x^2} h^2(x, t) P \tag{78.167}
\]

which takes the form

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} g(x, t) P + \frac{1}{2} \frac{\partial^2}{\partial x^2} h^2(x, t) P \tag{78.168}
\]

if (78.165) is interpreted as a stochastic differential equation in the Ito sense.

The two approaches have the following properties:

(i) in most of the models used in physics the Stratonovich
1. The system evolves according to the non-Hermitian Hamiltonian during the interval $dt$, or stochastic wave functions as follows. Starting from the master equation, one defines quantum trajectories, Wigner–Weisskopf theory \[78.17, 32\]. Here an initially mixed state and which interacts with the quantized electromagnetic field of free space. Even if none of the modes have a finite linewidth, called the natural linewidth.

2. The system makes a jump; i.e.,

$$|\psi(t + dt)\rangle = \frac{C|\psi(t)\rangle}{\sqrt{\langle \psi(t)|C^\dagger C|\psi(t)\rangle}}.$$  \(78.172\)

Since both possibilities describe a nonunitary time evolution, $|\psi\rangle$ must be normalized after each step. For each time interval $dt$ one of these two possibilities is randomly chosen according to the probability

$$P(t)dt = \langle \psi(t)|C^\dagger C|\psi(t)\rangle dt$$  \(78.173\)

to make a jump between $t$ and $t + dt$. We can now define a density operator

$$\rho_s(t) = |\psi(t)\rangle \langle \psi(t)|$$  \(78.174\)

for a specific quantum trajectory $|\psi(t)\rangle$. The density operator

$$\bar{\rho}_s(t) = |\psi(t)\rangle \langle \psi(t)|$$  \(78.175\)

averaged over all trajectories (indicated by the bar) is then a solution of the master equation (78.170).

This method can easily be generalized to master equations of the form (78.169).

### 78.12 Spontaneous Emission in Free Space

Consider an atom which is initially in one of its excited states and which interacts with the quantized electromagnetic field of free space. Even if none of the modes of the electromagnetic field is excited, there are still the vacuum fluctuations which "interact" with the atom and give rise to important effects:

1. **Spontaneous emission**: the atom spontaneously emits a photon and decays from the excited state.
2. **Natural linewidth**: due to the finite lifetime of the atomic levels, the radiation from an atomic transition has a finite linewidth, called the natural linewidth.
3. **Lamb shift**: the energy levels of the atom are shifted.

The standard theory of spontaneous emission is the Wigner–Weisskopf theory [78.17, 32]. Here an initially excited atomic state $|\ell\rangle$ decays exponentially according to

$$\langle \ell|\ell\rangle\frac{dt}{\Gamma_{\ell}} \rightarrow \langle \ell|\ell\rangle = e^{-\Gamma_{\ell} t}.$$  \(78.176\)

where the decay constant $\Gamma_{\ell}$ is given by

$$\Gamma_{\ell} = \sum_i \frac{\omega_{i\ell}^2 |d_{i\ell}|^2}{3\pi\epsilon_0 \hbar c^3}.$$  \(78.177\)

and the sum is over all atomic states with an energy $E_i$ lower than the energy $E_\ell$ of the state $|\ell\rangle$. $\omega_{i\ell} = (E_\ell - E_i)/\hbar$ is the transition frequency for the transition $|\ell\rangle \rightarrow |i\rangle$, and $d_{i\ell} = \langle \ell| \mu |i\rangle$ is the corresponding dipole moment.

The same decay constant $\Gamma_{\ell}$ is also observed as a linewidth in the spectrum of the radiation scattered...
by an atom when the incoming photon excites the atom to the level $|\ell\rangle$.

The energy level shift is more troublesome and needs the concept of mass renormalization, a standard problem in quantum electrodynamics. The theory and results are discussed in Chapt. 27 and Chapt. 28.

Recent calculations of Pachucki [78.45] based on fully relativistic quantum electrodynamics and including two-loop corrections predict 1 057 838(6) kHz for the energy difference between the $2s_{1/2}$-state and the $2p_{1/2}$-state which is in excellent agreement with the experimental result of 1 057 839(12) kHz [78.46].

For a discussion of energy levels and transition frequencies in hydrogen and deuterium atoms see also Sect. 28.3.

### 78.13 Resonance Fluorescence

Consider a two-level atom driven by a continuous monochromatic wave which is treated classically. The excited state of the atom can decay by spontaneous emission into vacuum modes of the electromagnetic field. This emission is called resonance fluorescence. Of particular interest are the properties of the emitted light. For a detailed discussion of resonance fluorescence, see for example [78.20, 31–33].

The far field at position $R$ emitted by an atom at the origin is proportional to its dipole moment and can be expressed in terms of the dipole operators $\sigma_+$ and $\sigma_-$ according to the relation [78.20]

\[
E^+(R, t) = -\frac{\alpha_0^2}{4\pi \epsilon_0 c^2 R^3} \sigma_-(t - r/c),
\]

\[
E^-(R, t) = -\frac{\alpha_0^2}{4\pi \epsilon_0 c^2 R^3} \sigma_+(t - r/c),
\]

where

\[
d = e \langle g | r | e \rangle
\]

is the atomic dipole matrix element and the field operators $E^+(R, t)$ and $E^-(R, t)$ as well as the dipole operators $\sigma_+(t)$ and $\sigma_-(t)$ are in the Heisenberg picture. Knowledge of the operators $\sigma_+(t)$ and $\sigma_-(t)$ is therefore sufficient to study the properties of the emitted light in the far field.

#### 78.13.1 Equations of Motion

The total Hamiltonian for the system reads

\[
H_{\text{tot}} = \frac{1}{2} \hbar \omega_0 \sigma_z + \sum_i \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right) + \hbar \sum_i \left( g_i \sigma_- b_i^\dagger + g_i^* \sigma_+ b_i \right) - \frac{1}{2} \left( d \, \epsilon^* \sigma_- e^{i\omega_0 t} + d^* \, \epsilon \, \sigma_+ e^{-i\omega_0 t} \right),
\]

where a resonant driving term has been added to the Hamiltonian (78.123). Here $d$ is the projection of the dipole matrix element $e \langle g | r | e \rangle$ onto the polarization vector of the driving field with an amplitude $\epsilon$. The corresponding master equation in the interaction picture is

\[
\dot{\rho} = -\frac{i}{\hbar} \Omega_1 \left[ \sigma_+ + \sigma_-, \rho \right] + \frac{1}{2} \left( 2\sigma_- \rho \sigma_+ - \sigma_+ \sigma_- \rho - \rho \sigma_+ \sigma_- \right),
\]

where $\Omega_1 = -\hbar d^* / \hbar$ is the Rabi frequency associated with the driving field. The vacuum modes of the field are described by a thermal reservoir at zero temperature.

The equations of motion for the expectation values $\langle \sigma_+ \rangle$, $\langle \sigma_- \rangle$, and $\langle \sigma_z \rangle$ are the optical Bloch equations with radiative damping (Chapt. 68) and are

\[
\frac{d}{dt} \langle \sigma_+ \rangle = -\gamma \langle \sigma_+ \rangle - i \frac{\Omega_1}{2} \langle \sigma_- \rangle,
\]

\[
\frac{d}{dt} \langle \sigma_- \rangle = -\gamma \langle \sigma_- \rangle + i \frac{\Omega_1}{2} \langle \sigma_+ \rangle,
\]

\[
\frac{d}{dt} \langle \sigma_z \rangle = -\gamma (\langle \sigma_z \rangle + 1) - i \Omega_1 \left( \langle \sigma_+ \rangle - \langle \sigma_- \rangle \right).
\]

These expectation values determine the density operator (78.142) of the two-level atom. Because (78.182) are a system of linear differential equations for $\langle \sigma_+ \rangle$, $\langle \sigma_- \rangle$, and $\langle \sigma_z \rangle$, they can be solved analytically. Furthermore, the quantum regression hypothesis allows one to calculate two-time correlation functions as shown in Sect. 78.9.

#### 78.13.2 Intensity of Emitted Light

According to (78.52) and (78.178), the intensity of the fluorescence light at position $R$ is given by

\[
I = \langle E^-(R, t) E^\dagger(R, t) \rangle \propto \langle \sigma_+ \sigma_- \rangle,
\]

and can be decomposed into two parts: the coherent intensity

\[
I_{\text{coh}} \propto \langle \sigma_+ \rangle \langle \sigma_- \rangle
\]
originating from the mean motion of the dipole, and the incoherent intensity

$$I_{\text{inc}} \propto \langle \sigma_+ \sigma_- \rangle - \langle \sigma_+ \rangle \langle \sigma_- \rangle, \quad (78.185)$$

which is due to fluctuations of the dipole motion around its average value. The steady state intensities are

$$I_{\text{coh}} \propto \frac{\Omega_1^2 \gamma^2}{(\gamma^2 + 2\Omega_1^2)^2}, \quad (78.186)$$

and

$$I_{\text{inc}} \propto \frac{2\Omega_1^4}{(\gamma^2 + 2\Omega_1^2)^2}. \quad (78.187)$$

For weak laser intensities ($\Omega_1$ small) the intensity of the fluorescence light is dominated by the coherent part whereas for high intensities ($\Omega_1$ large) it is dominated by the incoherent part.

### 78.13.3 Spectrum of the Fluorescence Light

The Wiener–Khintchine theorem (78.54) allows one to express the steady state spectrum of the fluorescence light as the Fourier transform of the correlation function $\langle \sigma_+ (\tau) \sigma_- (0) \rangle_{\text{ss}}$ in the form

$$S(\omega) = \frac{1}{2\pi} \int_0^\infty e^{-i\omega\tau} \langle E^{-} (\tau) E^{+} (0) \rangle_{\text{ss}} d\tau + \text{c.c.}$$

$$\propto \frac{1}{2\pi} \int_0^\infty e^{-(i\omega - \omega_0)\tau} \langle \sigma_+ (\tau) \sigma_- (0) \rangle_{\text{ss}} d\tau + \text{c.c.}$$

(78.188)

Again it consists of two contributions: a coherent part $S_{\text{coh}}(\omega)$, and an incoherent part $S_{\text{inc}}(\omega)$. The coherent part is

$$S_{\text{coh}}(\omega) \propto \frac{\Omega_1^2 \gamma^2}{(\gamma^2 + 2\Omega_1^2)^2} \delta(\omega - \omega_0). \quad (78.189)$$

The incoherent part of the fluorescence light has two qualitatively different spectra. For $\Omega_1 < \gamma/4$, it has a single peak at $\omega_0$, whereas it consists of three peaks for $\Omega_1 > \gamma/4$. For $\Omega_1 \gg \gamma/4$ it is given by

$$S_{\text{inc}}(\omega) \propto \frac{1}{2\pi\gamma} \left( \frac{\gamma/2)^2}{(\omega - \omega_0)^2 + (\gamma/2)^2} \right)$$

$$+ \frac{1}{3} \frac{(\omega - \omega_0 + \Omega_1)^2 + (3\gamma/4)^2}{(3\gamma/4)^2}$$

$$+ \frac{1}{3} \frac{(\omega - \omega_0 - \Omega_1)^2 + (3\gamma/4)^2}{(3\gamma/4)^2}. \quad (78.190)$$

The central peak at $\omega = \omega_0$ has a width of $\gamma/2$ whereas the width of the two side peaks at $\omega = \omega_0 \pm \Omega_1$ is $3\gamma/4$. Their heights are one third of the height of the central peak. This spectrum was predicted by Burshtein [78.47] and Mollow [78.48] and experimentally confirmed by Schuda et al. [78.49], Wir et al. [78.50], and Hartig et al. [78.51].

This triplet can be explained in terms of the dressed states $|1, n\rangle$ and $|2, n\rangle$ introduced in Chapt. 68 (78.50). If the driving field is resonant with the atomic transition, these states have the energies

$$E_{1,n} = \hbar \left(n + \frac{1}{2}\right) \omega_0 - \hbar \gamma_0,$$

$$E_{2,n} = \hbar \left(n + \frac{1}{2}\right) \omega_0 - \hbar \gamma_0. \quad (78.191)$$

(78.52). The energy differences between the allowed transitions are

$$E_{2,n} - E_{2,n-1} = \hbar \omega_0 + \hbar \gamma_0 - \hbar \gamma_1 \approx \hbar \omega_0,$$

$$E_{2,n} - E_{1,n-1} = \hbar \omega_0 + \hbar \gamma_0 + \hbar \gamma_1 \approx \hbar \omega_0 + \hbar \Omega_1,$$

$$E_{1,n} - E_{2,n-1} = \hbar \omega_0 - \hbar \gamma_0 - \hbar \gamma_1 \approx \hbar \omega_0 - \hbar \Omega_1,$$

$$E_{1,n} - E_{1,n-1} = \hbar \omega_0 + \hbar \gamma_0 - \hbar \gamma_1 \approx \hbar \omega_0. \quad (78.192)$$

where we have made the approximations

$$R_n - R_{n-1} \approx 0,$$

$$R_n + R_{n+1} \approx 2\gamma \sqrt{n + 1} \approx \Omega_1. \quad (78.193)$$

This is a good approximation for an intense driving field which can approximated by a highly excited coherent state with an average photon number $\bar{n}$.

Figure 78.6 shows these energy levels and the allowed transition. Obviously, the transitions correspond to frequencies $\omega_0$, $\omega_0 - \Omega_1$ and $\omega_0 + \Omega_1$.

---

**Fig. 78.6** Energy level diagram of dressed states. The transition frequencies are $\omega_0$, $\omega_0 - \Omega_1$ and $\omega_0 + \Omega_1$.
state picture also explains the 2:1 ratio for the integrated intensities of the central peak and the side peak in (78.190).

### 78.13.4 Photon Correlations

In addition to the spectrum which is based on the correlation function \(\langle E^{-} (\tau) E^{+} (0) \rangle_{ss}\) in Sect. 78.13.3, the second-order correlation function

\[
G^{(2)}_{ss}(\tau) = \langle E^{-} (0) E^{-} (\tau) E^{+} (\tau) E^{+} (0) \rangle_{ss}
\]

\[
\propto \langle \sigma_{\downarrow} (0) \sigma_{\downarrow} (\tau) \sigma_{\uparrow} (\tau) \sigma_{\uparrow} (0) \rangle_{ss}
\]

(78.194)

can be measured to gain more insight into the fluorescence light, see also Sect. 78.3.2. Experimentally this is done by measuring the joint probability for detecting a photon at time \(t = 0\) and a subsequent photon at time \(t = \tau\). Again the result can be obtained from the quantum regression hypothesis and reads

\[
g^{(2)}(\tau) = G^{(2)}_{ss}(\tau)/G^{(1)}_{ss}(0)^2
\]

\[= 1 - e^{-\gamma \tau} 4 \left( \cos \delta \tau + \frac{3\gamma}{4\delta} \sin \delta \tau \right),\]

(78.195)

where \(\delta\) is given by

\[
\delta = \sqrt{\Omega_{1}^2 - \gamma^2 / 4} .
\]

(78.196)

For \(\tau = 0\), \(g^{(2)}(0) = 0\), indicating a tendency of photons to be separated. This tendency is known as photon antibunching and was first predicted by Carmichael and Walls [78.52, 53] and experimentally verified by Kimble et al. [78.54, 55]. Photon antibunching of radiation emitted from a two-level atom has a simple explanation: After the atom has emitted a photon it is in the ground state and must first be excited again before it can emit another photon.

### 78.14 Recent Developments

This chapter has discussed the fundamentals of the quantized electromagnetic field and applications to the broad area of quantum optics. However, in the last eight years, quantum optics has blossomed in several new directions particularly in the key role it is playing in recent investigations of the fundamentals of quantum theory and related applications. In particular, the superposition principle (the bedrock of quantum mechanics), entanglement, the quantum-classical interface, and precision measurements have become very topical research areas, especially in respect to their relevance to quantum information processing.

#### 78.14.1 Literature

During the last eight years, several books on quantum optics [78.56–62] have been published. These books cover the topics of this chapter to some extent and take into account recent developments. For an introduction to the rapidly evolving fields of quantum information processing, we refer the reader to Chapt. 81 and [78.63–65].

#### 78.14.2 Field States

Recently, number states of the radiation field were observed in a cavity-QED experiment [78.66].

#### 78.14.3 Reservoir Theory

New research topics, such as quantum information processing, rely on the superposition principle and entangled quantum states. Since these states are very sensitive to decoherence, reservoir theory has attracted a lot of interest in recent years. Furthermore, as discussed in [78.67–72], decoherence is the physical process by which the classical world emerges from its quantum underpinning.

Many investigations in this area involve the presence of a reservoir (heat-bath/environment) and master equations are a ubiquitous tool. The familiar master equations of quantum optics are in Lindblad form [78.73], which guarantees that the density matrix is always positive definite during time evolution. In the derivation of this equation [78.74, 75], rapidly oscillating terms are omitted by the method of coarse-graining in time; the high frequencies correspond to the oscillator frequency \(\omega_{0}\) and, in the usual weak coupling limit, \(\omega_{0} \gg \gamma\), where \(\gamma\) is a typical decay constant. This is the rotating wave approximation (Sect. 66.3.2).

We have referred to the equations obtained prior to coarse-graining in time as pre-master (or pre-Lindblad) equations [78.74, 76], and such equations have been used extensively in other areas of physics [78.77, 78]; other authors have simply referred to them as master equations...
but, to avoid confusion, we reserve the latter term for equations in Lindblad form. Pre-master equations, like the master equations, describe an approach to the equilibrium state. This equilibrium state is the same in either case [78.76], but with pre-master (non-Lindblad) equations the approach can be through non-physical states of negative probability. However, as recently demonstrated, pre-master equations have other advantages vis-à-vis master equations:

(a) they lead to the exact expression for the mean value of $x(t)$ (as obtained from the exact Langevin equation for the problem);
(b) they lead, in the classical limit ($\hbar \to 0$), to the familiar Fokker–Planck equation of classical probability; and
(c) the exact master equation [78.79–83] is for long times of pre-master form. However, the general expectation (based on the time dependence of the coefficients) that the exact master equation preserves positivity for all times has not been realized since Ford and O’Connell have recently shown that, even in high temperature regime, the density matrix is not necessarily positive [78.84].

In traditional quantum optics, the emphasis has been on long-time ($t \gg \gamma^{-1}$) phenomena, for which the use of either master or pre-master equations is justified. However, they are both inadequate for dealing with short-time ($t \ll \gamma^{-1}$) phenomena (as can be shown most simply by calculating the mean-square displacement, a key ingredient in decoherence calculations), which are of much recent interest. Thus, it is desirable to use master equations. In that respect, the exact master equation of Hu et al. [78.79, 80] for an oscillator is an arbitrary dissipative environment has proved to be a popular and useful tool for which an exact solution has now been obtained [78.83]. However, it should also be mentioned that the solution of the initial value quantum Langevin equation gives all the same information as the exact master equation, and in fact, the solutions of the former were used to obtain the solutions of the latter [78.83].

The familiar two-level atom master equation is, of course, similar in form to the usual Lindblad-type master equation for the oscillator. However, motivated in particular by the desire to study decoherence and other short time phenomena, an exact master equation was derived [78.85] to study the non-Markovian dynamics of a two-level atom interacting with the electromagnetic field.

In addition, motivated by the desire to study a driven oscillator, the usual two-level atom master equation was generalized to include the case of an external force field [78.86, 87]. This generalized equation was then used not only to obtain the familiar zero-temperature Burshtein–Mollow spectrum, but also the corresponding high temperature results. For strong resonant driving at high temperature, the same three-peaked structure was observed in the zero temperature case, but a much larger width was found. The analysis, following other investigations, used the Lax formula for calculating two-time correlation functions. This formula is not a “quantum regression theorem” as it is often designated (see also Sect. 78.9), but simply an approximation (which more resembles an Onsager classical regression theorem [78.88]) which works very well in the case of weak coupling and for frequencies near a resonant frequency, but not otherwise [78.86, 87].

In Sect. 78.6, we stressed the usefulness of quasi-probability distributions instead of the density matrix, with particular attention to the Wigner distribution. In particular, for simple Hamiltonians, we pointed out that the equation for the corresponding Wigner function has the form of a Fokker–Planck equation and we considered the explicit form describing the usual master equation. The more general equations associated with an exact master equation and their solution was the subject of [78.83] and interesting limits of that equation, including the pre-master equation for both momentum coupling and coordinate coupling were discussed at length in [78.89, 90]. In the case of two-level systems, it is not convenient to use quasi-probability distributions; instead, it is found that the preferred tool is the polarization vector [78.91]. Surprisingly, it has not been generally adopted by the quantum optics community although its usefulness in that context has been demonstrated recently in [78.86, 87].

References

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