Independent Oscillator Model of a Heat Bath: Exact Diagonalization of the Hamiltonian

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The problem of a quantum oscillator coupled to an independent-oscillator model of a heat bath is discussed. The transformation to normal coordinates is explicitly constructed using the method of Ullersma. With this transformation, an alternative derivation of an exact formula for the oscillator free energy is constructed. The various contributions to the oscillator energy are calculated, with the aim of further understanding this formula. Finally, the limitations of linear coupling models, such as that used by Ullersma, are discussed in the form of some critical remarks.

KEY WORDS: Coupled oscillators; heat bath; free energy; quantum dissipation

1. INTRODUCTION

In recent publications we have shown that the quantum Langevin equation affords a powerful and physically appealing approach to the problem of an atom in a blackbody radiation field. This problem goes back to van Kampen's thesis, where a nonrelativistic atom interacting via dipole coupling with the electromagnetic field was first discussed with modern methods. There, too, he first pointed out that the problem of a harmonically bound electron (oscillator) is exactly soluble and provides an instructive model for real atoms. Some years later, in a classic series of four papers, his student P. Ullersma discussed the general model of an
The normal mode equations are of the form:

\[
\frac{\rho - \varepsilon_m}{\varepsilon_0} \left( \frac{1}{m^2} \sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \right) = \lambda_x - \varepsilon_m
\]

(2.8)

Putting this in (2.7), that equation becomes

\[
(\varepsilon_0)^\rho \left( \frac{\rho + \varepsilon_0}{\varepsilon_0} \sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \right) = \lambda_x
\]

Putting (2.8) in (2.7), that equation becomes

(2.9)

It follows from (2.6) that

(2.10)

The equations of motion then become

(2.11)

We seek normal mode solutions of the form

(2.12)

The equations of motion according to the Lagrangian (2.1) take the form

2. THE ID MODEL

2.1. Normal Modes

Independent Oscillator Model

The normal mode equations are of the form

\[
\frac{\rho - \varepsilon_m}{\varepsilon_0} \left( \frac{1}{m^2} \sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \right) = \lambda_x - \varepsilon_m
\]

(2.1)

and

\[
\frac{\rho - \varepsilon_m}{\varepsilon_0} \left( \frac{1}{m^2} \sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \right) = \lambda_x
\]

(2.2)

and

\[
[\varepsilon - \lambda_x] \left( \frac{1}{m^2} \sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \right) = \lambda_x
\]

(2.3)

and

\[
\sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \lambda_x = \lambda_x
\]

(2.4)

and

\[
\sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \lambda_x = \lambda_x
\]

(2.5)

and

\[
\sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \lambda_x = \lambda_x
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\]

(2.12)

and

\[
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\]

(2.13)

and

\[
\sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \lambda_x = \lambda_x
\]

(2.14)

and

\[
\sum_{n=1}^{N} [\gamma_{mn}^2 - (\varepsilon_0 + \varepsilon_m)] \lambda_x = \lambda_x
\]
\[ N^{\cdot \cdot \cdot} x = 0, \quad \dot{x}^{\cdot \cdot \cdot} + \dot{x}^3 = 0 \]

We require the normalization condition

\[ N^{\cdot \cdot \cdot} x = 1 \]

and, using (217)

\[ \left( \frac{\partial}{\partial \xi} \right) x = -x \]

We now consider the matrix of transformation to normal co-ordinates.

We use the matrix of transformation to normal co-ordinates

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

where

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

and

\[ \left( \frac{\partial^2}{\partial \xi^2} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial^2}{\partial \xi^2} \right) x^n \]

Applying (217) to the equation of motion, we get

\[ \ddot{x} + x = 0 \]

\[ \Rightarrow \left( \frac{\partial}{\partial \xi} \right) x = -x \]

Introducing (216)

\[ \left( \frac{\partial}{\partial \xi} \right) x = -x \]

Finally, we can write

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

where

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

We also write

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

where

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

The solution of the resulting equations of motion can be written

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]

The number of distinct normal mode frequencies is one more than

\[ \left( \frac{\partial}{\partial \xi} \right) x = \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \xi} \right) x^n \]
We introduce normal mode coordinates through the relation

\[ \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = 0 \]

and

\[ \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = 0 \]

for all \( n \). Therefore, we can write the orthogonality relation for the transformation \( T \) as

\[ T x = \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n \]

which is equivalent to the completeness relation (2.17). The goal of the completeness relation (2.17) is to show that the transformation (2.17) is valid. Here the second is a simple identity which is valid in general

\[ \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = 0 \]

The transformation is valid if and only if

\[ \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = 0 \]

The inverse of this transformation is

\[ \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = \sum_{n=0}^{\infty} X_n \frac{\partial^2}{\partial \tau^2} x_n = 0 \]
the contour C encloses all the poles of the integrand, as shown in Fig. 2. We now deform the contour into the contour D, where we have enclosed all the poles of $(e^{2πi/3} + e^{-2πi/3})$.

\[
(\nu_C x)^\nu \left[ \frac{(\nu_C z - 1) + \nu C}{\nu - 1} \right] \left( \nu C^2 z \right)^{\nu - 1} = (\nu C)^{\nu - 1}
\]

As we have noted, this is the result of the pole of $\nu_C$ at $z = 0$.

(3.12) \[
\left( \int \frac{x^n}{(x+a)^n} \right) \left( \frac{x-a}{x+b} \right) \left( \frac{x-c}{x-d} \right) = (\nu C)^{\nu - 1}
\]

We can rewrite the expression for $\nu C$ in terms of the contour $\nu D$ using the two expressions (2.9) and (2.11).

The expression $\nu C$ is the counterpart to the expression $\nu D$ for the contour in the complex plane. We have a similar formula for the contour $\nu C$. However, when we deform the contour $\nu C$ to the contour $\nu D$, we obtain a different result.

\[
(\nu C)^{\nu} - (\nu D)^{\nu} = (\nu C)^{\nu - 1}
\]

(3.3) \[
(\nu C) = \left( \begin{array}{c} \frac{1}{N-1} \\ \nu C \end{array} \right)
\]

This is the Hermitian form of the Hamiltonian (3.1).

(3.4) \[
\frac{1}{N-1} \nu C = H
\]

(3.5) \[
\left( \begin{array}{c} \nu C \end{array} \right) = H
\]

(3.6) \[
\left( \begin{array}{c} \nu C \end{array} \right) = H
\]

This is the Hermitian form of the Hamiltonian (3.1).

(3.7) \[
\left( \begin{array}{c} \nu C \end{array} \right) = H
\]

The Hamiltonian operator for the oscillation is obtained by applying the Hermitian form to the Hamiltonian equation (3.1).

(3.8) \[
\left( \begin{array}{c} \nu C \end{array} \right) = H
\]

This is the Hermitian form of the Hamiltonian (3.1).
This is deferred to be.

For completeness we give the formula for the oscillator heat energy:

\[
\frac{1}{(\pm 0 + \infty)\pi} \int_{-\infty}^{\infty} e^{i\omega t} (L_{\infty}^m n) e^{\frac{-i}{\hbar} p} = (L_{\infty}^m n)
\]

Therefore, we may write

\[
\pm \infty^{(\pm 0 + \infty)\pi} = (\pm 0 + \infty)\pi
\]

This formula can be simplified somewhat if we note that \(\pm 0 + \infty\) is an odd

\[
\frac{1}{(\pm 0 + \infty)\pi} \int_{-\infty}^{\infty} e^{i\omega t} (L_{\infty}^m n) e^{\frac{-i}{\hbar} p} = (L_{\infty}^m n)
\]

Therefore, we may write

\[
\frac{z^p}{(z^p e^{-z^p} e^{\frac{-i}{\hbar} p} = (L_{\infty}^m n)
\]

We see that the oscillator energy (11.5) can be written (11.4) and recognizing the sum as \(e^{i\pi n/2}\) we

\[
\frac{1}{(\pm 0 + \infty)\pi} \int_{-\infty}^{\infty} e^{i\omega t} (L_{\infty}^m n) e^{\frac{-i}{\hbar} p} = (L_{\infty}^m n)
\]
Here we have dropped the principal value prescription, since

\[ \int_{-\infty}^{\infty} (\frac{\phi}{\phi}) d\phi = \langle \phi \rangle \]

and we obtain the result

\[ \langle \phi \rangle = \int_{-\infty}^{\infty} (\frac{\phi}{\phi}) d\phi = \langle \phi \rangle \]

where \( \langle \phi \rangle \) is the expectation value of \( \phi \).

Here we have used the summation convention, and we write \( \sum_{n=1}^{N} \frac{1}{\lambda_n} \) to denote the total energy. The Hamiltonian \( H \) is the interaction matrix in the basis \( \left\{ \phi_1, \phi_2, \ldots, \phi_N \right\} \).

Here is the Hamiltonian for the free oscillator, \( H = \hbar \omega \).

\[ \hbar \omega \sum_{n=1}^{N} \frac{1}{\lambda_n} = \hbar \omega \]

where \( \hbar \) is Planck's constant.

4. Contributions to the mean energy

\[ \langle H \rangle = \langle \phi \rangle \omega \]

where \( \langle \phi \rangle \) is the average energy of the system.
\[
\left( \frac{\partial_f}{\partial_t} \right) \left[ \left( \frac{\partial_f}{\partial_f} \right) + \left( \frac{\partial_f}{\partial_t} \right) \right] = \left( \frac{\partial_f}{\partial_f} \right) - \left( \frac{\partial_f}{\partial_t} \right)
\]

where in place of \( \phi(\tau) \).

\[
\left( \phi(\tau) + \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

The expression of the Hamiltonian terms in (1) can be derived by

\[
\left( \phi(\tau) + \phi(\tau) \right) + \left( \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

Since the Hamiltonian terms in (1) are used in the form of linear combinations, the terms in (1) can be expressed as a linear combination of the Hamiltonian terms.

\[
\left( \phi(\tau) + \phi(\tau) \right) + \left( \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

(5) In the case of the Hamiltonian, the expression is essentially identical to the case where the Hamiltonian is expressed as a linear combination of the Hamiltonian terms. Therefore, we have:

\[
\left( \phi(\tau) + \phi(\tau) \right) + \left( \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

(6) Consider now the expression of the Hamiltonian:

\[
\left( \phi(\tau) + \phi(\tau) \right) + \left( \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

(7) Consider now the expression of the Hamiltonian:

\[
\left( \phi(\tau) + \phi(\tau) \right) + \left( \phi(\tau) \right) = \left( \phi(\tau) \right)
\]

One might be tempted to identify this expression with the Hamiltonian, taking the methods illustrated above we find the Hamiltonian terms in (1) are used in the form of linear combinations, the terms in (1) can be expressed as a linear combination of the Hamiltonian terms.
REFERENCES

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Acknowledgments

(4.11) 

$$p = 1 - \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(4.12) 

$$(\zeta + m - \omega)$$

As in (4.10), from this we get that