Attractive spin-spin contact interactions in the Einstein-Cartan-Sciama-Kibble
torsion theory of gravitation

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Kerlick has obtained the unexpected result that the spin-spin contact interactions, which are characteristic
of the Einstein-Cartan-Sciama-Kibble theory of gravitation, are attractive for the case of a totally
antisymmetric spin angular momentum density \( \tau_{ij} \), which, in particular, is appropriate for the Dirac field.

Using our previous techniques—where the emphasis is on the use of Lagrangian densities as opposed to
energy-momentum densities—we present a simple and explicit verification of this result.

A characteristic feature of the Einstein-Cartan-Sciama-Kibble (ECKS) theory of gravitation\(^1\) is the
appearance of spin-spin contact (SSC) interactions. In a recent analysis we compared such interac-
tions with contact interactions which arise in a quantum version of Einstein's theory. Our choice
for the spin angular momentum density \( \tau_{ij} \) was

\[
\tau_{ij} = \epsilon_{ijmn} U^m U^n S^m,
\]

which led to a repulsive contribution to the gravitational interaction from the SSC terms in the ECKS
theory. However, we also cautioned\(^2\) that different interactions may arise from the use of spin
densities different from that given in Eq. (1). In fact, Kerlick\(^3\) has shown that the SSC terms for a
Dirac field actually enhance the attractive nature of the gravitational interaction, whereas the op-
posite result (repulsive contribution) is obtained for a semiclassical spinning fluid. This arises
from the fact that \( \tau_{ij} \) is totally antisymmetric for the Dirac field. Now Kerlick's analysis was based
on the use of energy-momentum tensors, which of course were necessary for his discussion of cos-
mological models. Here we point out that, if one's attention is confined to an analysis of the basic
nature of the interaction, the simplest approach is via the use of a Lagrangian density.

As in Ref. 1, our starting point is the non-Riemannian contribution to the total Lagrangian, viz.,

\[
\Delta \mathcal{L} = \hat{k} (-\frac{1}{2} \tau_{ijk} \tau^{ijk} + \tau_{ij} \tau^{ij} + \tau_{ik} \tau^{ik}),
\]

where \( \hat{k} = 8\pi G/c^4 \).

In the case where \( \tau_{ij} \) is totally antisymmetric,

it follows immediately that

\[
\Delta \mathcal{L} = \frac{1}{2} \hat{k} \tau_{ijk} \tau^{ijk}.
\]

As with Kerlick\(^4\) we now write

\[
\tau_{ijk} = \epsilon_{ijk\ell} \tau^\ell,
\]

where

\[
\tau^\ell = \frac{1}{4} \bar{\psi} \gamma_{\ell} \psi.
\]

In the "rest" picture, we can write\(^6\)

\[
\tau^\ell = \frac{1}{4} (\bar{\psi} \gamma^\ell \psi) = \frac{1}{2} S^\ell.
\]

It immediately follows, from Eqs. (3), (4), and (6) that

\[
\Delta \mathcal{L} = \frac{3}{4} \hat{k} S^2.
\]

In other words, the SSC interaction is attractive in the case of a field whose spin angular momen-
tum density is totally antisymmetric. This should be contrasted with the result (\( \Delta \mathcal{L} = -\hat{k} S^2 \))

obtained with the choice of \( \tau_{ij} \) given by Eq. (1).

Comparing \( \Delta \mathcal{L} \) with the corresponding term which occurs in a quantum version of Einstein's
theory, viz., \( \Delta \mathcal{L}^{(1)} \) (see Ref. 1) we obtain

\[
\Delta \mathcal{L}^{(1)} = -\frac{3}{8} \Delta \mathcal{L}.
\]

Thus, the overall contribution of both spin contact terms is to give an attractive gravitational
effect. Of course, repulsive gravitational forces of a different nature\(^1\) are still present.

The author is pleased to acknowledge the stimulation he received to look further into such ques-
tions as the result of a discussion with Professor A. Trautman.

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\(^1\) R. F. O'Connell, Phys. Rev. Lett. 37, 1653 (1976); 38, 298(E) (1977) contains relevant references. Also, for
the most part, we use the notation of this paper.

\(^2\) Reference 11 of Ref. 1.


\(^5\) J. J. Sakurai, Advanced Quantum Mechanics (Addison-