Bloch oscillations in small-capacitance Josephson junctions

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An analytic study of ac-bias-current effects on the Bloch oscillations in small Josephson junctions at the $E_c \gg E_J$ limit ($E_c$ is the charging energy and $E_J$ is the Josephson energy) is presented. By solving the quasicharge differential equation proposed by Likharev et al., solutions for the quasicharge $q(t)$, as well as the $I$-$V$ curves, are analyzed. It is shown that due to the sudden jump of the Bloch-oscillation period at some value of the dc current $I$, resistive steps of anomalous differential resistance emerge in the $I$-$V$ curve. A general relation between the position of the resistive step on the $I$-$V$ curve and the applied frequency $f$ is derived. The connections and the differences between our formula and the weak-damping-limit formula $I = (m/n)2ef$ are discussed. The results are compared with recent experiments.

I. INTRODUCTION

The study of Bloch oscillations in small-capacitance Josephson junctions has attracted much theoretical and experimental interest in recent years. It is predicted that Bloch oscillations in solids occur when a constant electric field is applied, where an electron of momentum $\pi \hbar/a$ (a is the lattice constant of the solid) acquired from the field, may reverse its motion by giving a momentum of $2\pi \hbar/a$ to the lattice. Similar Bloch oscillations are predicted for a small Josephson junction of capacitance $C_J$, such that the charging energy $E_c = e^2/2C_J > k_B T$ ($T$ is the temperature) and such that the band character of the energy spectrum is prominent. This kind of Bloch oscillation is described as "the process of a periodic discrete transfer of a Cooper pair between the junction electrodes interposed by a recharging of the junction capacitance by the external current." In particular, in the case of a constant current $I$, it is predicted: (i) in the absence of damping the Bloch-oscillation frequency $f_B = I/2e$; (ii) the addition of an ac part, of frequency $f$, to the junction current results in voltage steps in the dc current-voltage ($I$-$V$) curves, located at $I = (m/n)2ef$, where $m$ and $n$ are integers.

Many theoretical methods have been used to study Bloch oscillations in small Josephson junctions. Among them, the quantum-Langevin-equation (QLE) approach is perhaps the simplest one. In this approach, a simple Langevin-type equation for the junction quasicharge $q(-e \leq q \leq e)$ is derived in a single band approximation, where the quasicharge differs from the real charge of the junction in exactly the same way that the quasimomentum of an electron in a crystal lattice differs from its real momentum. When the current fluctuations are small and can be neglected, a first-order differential equation for the quasicharge is deduced and solved numerically (see Fig. 5 in Ref. 1). The results of this numerical study form the basis for the quantitative predictions of Bloch oscillations. The purpose of this paper is to present an analytic solution to the same problem. Then, we use this result to analyze in a tractable manner the Bloch oscillations, the $I$-$V$ curves, and the resistive steps. While many of our results agree with the numerical results obtained in Ref. 1, we also find that some important differences exist.

In Sec. II, we consider the equation obeyed by the quasicharge in a small Josephson junction, with a bias current consisting of both dc and ac parts. We show, in the limit $E_c \gg E_J$, an analytic solution may be obtained. Also, in the absence of the ac current bias, our general results are shown to reduce to known results appearing in the literature. In Sec. III, we analyze the ac effects on the Bloch oscillations in the weak-damping limit. The more general case of arbitrary damping is treated in Sec. IV, with emphasis placed on a determination of $I$-$V$ curves and the Bloch-oscillation frequency. In Sec. V we discuss our conclusions and their possible relevance to a recent experimental study.

II. ANALYTIC SOLUTION OF THE QUASICHARGE EQUATION OF MOTION

Consider a small-capacitance Josephson junction shunted by a weak ohmic resistor (with resistance $R_J \gg R_Q \equiv h/4e^2$) subjecting to the following conditions: (i) the voltage across the junction is much less than the energy gap in the superconducting electrodes of the junction; (ii) the temperature is much lower than the gap between the ground state and the first excited state of the energy bands. It is shown in Ref. 1 that in a single band approximation the quasicharge $q$ evolves according to a Langevin equation

$$\frac{dq}{dt} + \frac{V^{(0)}}{R_J} = I(t) + I_F(t), \quad -e \leq q \leq e,$$

where $R_J$ is the shunt resistance, $I(t)$ is the bias current, and $I_F(t)$ is the fluctuation current, and

$$V^{(0)} = dE^{(0)}/dq,$$

with $E^{(0)}$ the ground-state energy. Previously, the $I$-$V$ curves of the system were obtained in Ref. 1 by numerically studying (1) for particular values of $E_c/E_J$, at $I_F = 0$ and

$$I(t) = I + I_A \cos \omega t,$$
where \( I \) is the dc current, and \( I_A \) is the amplitude of the ac current bias. However, in the \( E_c \gg E_J \) limit we can solve the problem analytically, as we will now demonstrate.

In the \( E_c \gg E_J \) limit,\(^1\) the \( V^{(0)} \) of (2) takes a simple form \( q/C_J \), and thus, when the noise effect is negligible, Eqs. (1)–(3) reduce to

\[
\frac{dq}{dt} = I + I_A \cos \omega t, \quad -e \leq q \leq e, \quad \ldots \quad (4)
\]

where \( \tau_J = C_J R_J \). Equation (4) is a first-order linear differential equation, the solution of which can be readily obtained as

\[
q(t) = \tau_J \left[ I - \left( I - \frac{q(0)}{\tau_J} + I_A \sin^2 \theta \right) e^{-t/\tau_J} \right.
\]

\[
+ I_A \sin \theta \sin(\theta + \omega t) \left. \right] , \quad \ldots \quad (5)
\]

where

\[
\sin \theta = (\omega^2 \tau_J^2 + 1)^{-1/2}. \quad \ldots \quad (6)
\]

Equation (5) is a key result and it will form the basis of our further analysis. We now adopt the physical picture\(^1,2\) that the Bloch wave oscillations for Josephson junctions consist of a periodic discrete transfer of Cooper pairs between the junction electrodes, interposed by a recharging of the junction capacitance by the external current. Defining the Bloch-oscillation period \( t_B \) as the time during which \( q(t) \) changes from \(-e\) to \(+e\), and taking \( q(0) = -e \), and it follows that \( q(t_B) = e \). Hence, the exact value of \( t_B \) can be obtained from (5) with the result

\[
\frac{I}{(e/\tau_J)} = \coth \frac{I_B}{2 \tau_J} + \frac{I_A}{(e/\tau_J)} \frac{\sin \theta [\sin \theta e^{-t_B/\tau_J} - \sin(\theta + \omega t_B)]}{1 - e^{-t_B/\tau_J}} . \quad \ldots \quad (7)
\]

Equation (7) is our second key result since it provides the basis for an analytic evaluation of the Bloch frequency. Since the system undergoes Bloch oscillations, the average of a physical quantity is evaluated over the time interval \( t_B \). Thus we obtain for the averaged voltage

\[
V = \frac{1}{C_J} \langle q(t) \rangle = \frac{1}{C_J t_B} \int_0^{t_B} q(t) dt
\]

\[
= R_J \left[ I - \theta (I - I_A) \frac{2e}{t_B} - I_A \sin \omega t_B \right] . \quad \ldots \quad (8)
\]

where the last equality follows by use of (4). This is our third key result. Here \( \theta \) is the Heaviside step function, \( t_B \) is determined by (7), and \( I_T \) is a threshold current (the magnitude of \( I_T \) will be discussed later) below which (7) does not have a solution. What happens is that if \( I \) is smaller than \( I_T \), then the current dissipates so fast through the resistance \( R_J \) that it is not sufficient to build up the charge on the capacitance from \(-e\) to \(+e\).

The coupled equations (7) and (8) are the solutions for the current and voltage of Bloch oscillations in small Josephson junctions in the framework of the quasicharge equation (4). When \( I_T = 0 \), (7) and (8) reduce, respectively, to the well-known results\(^2\)

\[
t_B = \tau_J \ln \frac{I + I_0}{I - I_0} , \quad \ldots \quad (9)
\]

\[
V = R_J \left[ I - \theta (I - I_0) \frac{2e}{t_B} \right] , \quad \ldots \quad (10)
\]

where \( I_0 \equiv (e/\tau_J) \) is the threshold current below which there is no Bloch oscillation (the superscript zero denotes the dc case). Equations (9) and (10) show that for a small Josephson junction with a dc bias (see the \( I_A = 0 \) curve in Figs. 1 and 3), below a threshold value \( I_0 \) the current is flowing completely through the shunt and the voltage is ohmic \( (V = I R_J) \). For \( I > I_0 \), one finds a crossover to Bloch oscillations, where the dc voltage decreases with the increasing current. The Bloch oscillation frequency is deduced from (10) as

\[
f_B = \frac{1}{t_B} = \frac{1}{2e} \left[ I - \frac{V}{R_J} \right] , \quad \ldots \quad (11)
\]

in agreement with Refs. 1 and 2. Also, referring to the \( I_A = 0 \) curve in Fig. 3, we note that the \( I-V \) curve is smooth except at the threshold point \( I = I_0 \), where there is a resistive step, i.e., the differential resistance \( dV/dI \) switches abruptly from positive to negative values.

III. BIAS CURRENT WITH dc AND ac COMPONENTS: WEAK-DAMPING LIMIT

In this section we analyze our main results (5), (7), and (8) in the weak-damping limit \( (R_J \rightarrow \infty) \), where the mathematics is much simpler and where the physics is more transparent.

When \( R_J \rightarrow \infty \) \( (\tau_J \rightarrow \infty) \), the quasicharge solution (5) reduces to

\[
q(t) = I t + \frac{I_A}{\omega} \sin \omega t + q(0) . \quad \ldots \quad (12)
\]

Structurally, (12) is similar to the well-known form\(^6\) for the phase difference \( \phi(t) \) of a Josephson junction with an applied voltage \( V(t) = V + V_A \cos \omega t \);

\[
\phi(t) = \frac{2e}{h} V t + \frac{2e V_A}{h \omega} \sin \omega t + \phi(0) . \quad \ldots \quad (13)
\]

We note that (13) gives us an explicit expression, in terms of the parameters of the applied voltage, for the Josephson current \( J(t) = J_c \sin \phi(t) \), where \( J_c \) is the Josephson critical current. It is found\(^6\) that whenever the relation \( 2e(V/h) = n \omega \), where \( n \) is an integer, is satisfied, a dc current component is present in the Josephson current
This dc current component results in current steps in the $I\!-\!V$ curves, and agrees very well with the experimentally found Shapiro steps.\textsuperscript{4,6}

The similarity between the expressions (12) and (13) for the dynamical variables for the Bloch oscillations and the phase difference for the Josephson oscillations, respectively, and the connection between (13) and the Shapiro steps, led some researchers\textsuperscript{1,4} to conclude that (12) will result in voltage steps whenever $\pi I/e = n\omega$, or, equivalently $I = 2enf$ where $f = \omega/2\pi$. Here we argue that the above speculation is not true mainly due to the following two reasons: First, there is an essential difference between the $q(t)$ of (12) and the $\phi(t)$ of (13) in that $q(t)$ is a compact variable defined by $-e \leq q(t) \leq e$, whereas $\phi(t)$ is an extended variable whose value is unrestricted. To state it another way, if $q(t)$ experiences periodic motion between $-e$ and $e$ with a period $t_B$, it follows that the $t$ in (12) is restricted to be between zero and $t_B$, in contrast to the fact that the $t$ appearing in (13) is unrestricted. Our second reason against an exact quantum analogy between (12) and (13) is that the corresponding features of the $I\!-\!V$ curves for (12) at $\pi I/e = n\omega$ and (13) at $2eV/\hbar = n\omega$ are totally different. As we stated earlier, in the case of Josephson oscillations, the relation $2eV/\hbar = n\omega$ induces Shapiro current steps in the $I\!-\!V$ curves. As for the Bloch oscillations, as we will now demonstrate there are no corresponding dc current steps.

To obtain the $I\!-\!V$ curve for the Bloch oscillations, we first of all use (12) and the fact that $q(t_B) - q(0) = 2e$ to obtain

$$I = \frac{2e}{t_B} - \frac{I_A}{\omega t_B} \sin\omega t_B,$$

(a result which also follows from (7) by taking the $\tau_J \to \infty$ limit. Also, from (12) and the general relation for $V$ given in the first line of (8), we obtain

$$V = \frac{I_A}{C_J \omega} \left[ 1 - \cos\omega t_B - \frac{1}{2} \sin\omega t_B \right].$$

Thus, for every given value of $I_A$, $I$, and $\omega$, we calculate $t_B$ from (14) which is then used to calculate $V$ from (15).

Using (14) and (15), we have carried out numerical calculations which show that the $I\!-\!V$ curves do not display voltage steps at the points $\pi I/e = n\omega$. As a result, we conclude that there is no analogy to Shapiro steps in the arena of Bloch oscillations.

### IV. BIAS CURRENT WITH dc AND ac COMPONENTS: $I\!-\!V$ CURVES AND BLOCH-OSCILLATIONS FREQUENCIES

Having shown in the last section that the simple analogy between the ac effects on Josephson oscillations and Bloch oscillations is not valid, in this section we study the ac effects on Bloch oscillations in a more detailed manner. In particular, we do not restrict ourselves to the weak-damping limit. Our approach is based on Eqs. (4)–(8).

A. Bloch-oscillation period

In the presence of the ac-bias current ($I_A \neq 0$), from (7) one observes that the dc-bias current $I$ becomes an oscillating function of the Bloch oscillating period $t_B$. When $t_B / \tau_J \gg 1$, we see from (7) that $I = t_B^0 - I_A \sin\theta \sin(\theta + \omega t_B)$, i.e., $I$ has a period of $2\pi/\omega$ and an amplitude of $I_A \sin\theta$. This is illustrated in Fig. 1, where we plot (7) ($I/I_B^0$ vs $t_B / \tau_J$) for $\omega \tau_J = \pi$ and $I_A/I_B^0 = 0$ and 1.0. The minimum values of $I$ are designated as $I_n$ ($n = 1, 2, 3, \ldots$) in order of decreasing value. As indicated by Fig. 1, for each value of $I$ there may exist a set of $t_B$ which satisfies (7). Since at a fixed $I$ the shortest $t_B$ process has the best chance to recharge the system from $-e$ to $e$ before something else happens, the high-value solutions of $t_B$ are nonphysical. This is indicated by Fig. 1, where the high-value solutions of $t_B$ are drawn by dashed curves, and the actual unique physical solution of $I$ vs $t_B$ arising from (7) is drawn by full curves. The “physical” value of $t_B$ corresponding to the current $I_n$ is designated as $t_{Bn}$. The full curves in Fig. 1 show that at fixed $I_A$ and $\omega$, $I$ is a decreasing function of $t_B$ with periodic downward steps at $I = I_n$, where $t_B$ jumps abruptly from $t_{Bn}$ to $t_{Bn} + \Delta$, and $\Delta$ is approximately $2\pi/\omega$. The lowest possi-

![FIG. 1. dc current bias $I$ (in units of the threshold current $I_B^0$) as a function of the Bloch-oscillation period $t_B$ (in units of the characteristic time $\tau_J$ of the junction) for a small-capacitance Josephson junction with additional ac-bias current $I_A \cos \omega t$. Full curves are the physical unique solutions of (7) calculated at $\omega \tau_J = \pi$ and $I_A/I_B^0 = 1.0$.](image-url)
ble value of $I_A(n \to \infty)$ corresponds to the threshold current $I_t$ below which there is no Bloch oscillation. By looking at the minimum value for the $I$ of (7) in the $t_B \to \infty$ limit, we obtain for general values of $I_A$ and $\omega\tau_j$, the threshold current

$$I_t = I_n - I_{A}\sin \theta.$$  \hfill (16)

The appearance of the jump of the Bloch-oscillation period (the $t_{Bn}$ steps) seen in Fig. 1 at some particular values of $I_n$ deserves more discussion. From a physical point of view, at $I > I_t$, there are two main factors, the time interval $\Delta (\sim 2\pi/\omega)$ between consecutive values of $t_{Bn}$ and the magnitude of $I_{Bn}$, which determine the experimental significance of the $t_{Bn}$ steps seen in Fig. 1. As a measure of the width of the steps, the importance of $2\pi/\omega$ is quite obvious. The actual value of $t_{Bn}$ is also crucial because other physical processes may obliterate the Bloch oscillations, if $t_{Bn}$ is too large. In any case, it is clear that a large $2\pi/\omega$ and a small $t_{Bn}$ will favor the detection of the $t_{Bn}$ steps. Bearing this point in mind, we now discuss the $t_{Bn}$ steps in more detail.

The other equation which can be used to solve for the exact value of $I_n$ and $t_{Bn}$ for any given integer $n$ can be obtained by studying the minimum conditions of (7): $dI/dt_B = 0, d^2I/dt_B^2 > 0$. After some algebra, from these minimum conditions of (7), we obtain a remarkably simple relation between $t_{Bn}$ and $I_n$:

$$\omega t_{Bn} = 2n\pi + \cos^{-1} \frac{I_{t}^0 - I_{n}}{I_{A}}, \quad n = 0, 1, 2, \ldots.$$  \hfill (17)

For given values of $I_A$ and $\omega$, the values of $I_n$ and $t_{Bn}$ can now be obtained by solving the coupled equations (7) and (17). We note that depending on the actual values selected for $I_A$ and $\omega$, (13) may not be satisfied for all $n$ values. In fact, it turns out that there is a lowest possible value for $n$ for which (17) is satisfied. Figure 1 is an example of this feature since there is no minimum for the current at $\omega t_B$ values less than $2\pi$, i.e., we have no $I_0$ nor $t_{B0}$. Instead, the first minimum occurs at a value of $\omega t_B$ between $2\pi$ and $4\pi$ and this is designated as $\omega t_{B1}$ [according to (17)] with $I_1$ denoting the corresponding current minimum.

Equation (17) is a very striking result. First, it shows that at the current $I_n$, $\omega t_{Bn}$ takes restricted values between $2n\pi + \pi/2 - \theta$ and $2n\pi + \pi$ where the former value is obtained by using (16). This indicates that $\omega t_{Bn}$ cannot take the value of $2n\pi$, whereas the value of $\omega t_{B} t_B$ is not restricted. The difference between $\omega t_{Bn}$ and $\omega t_B$ becomes very important later when we study the $I$-$V$ curve for the system.

Second, the smallest possible (optimum) value for observing the period of the Bloch oscillations can be determined in the following way: Since (16) and (17) imply, respectively, that $I = I_t^0 - I_A\sin \theta \leq I_n$, and $I_t^0 - I_A \leq I_1 \leq I_t^0 + I_A$, it follows that $I_t^0 - I_A \sin \theta \leq I_1 \leq I_t^0 + I_A$, which gives us information on the actual regions where "steps" occur. For fixed values of $I_A$ and $\omega$, $I_t$ is a decreasing function of $t_{Bn}$ (see Fig. 1, for example). Thus, if there exists a solution of (17) for the maximum current $I_t = I_t^0 + I_A$, then $t_{Bn}$ takes the optimum (smallest) value, $t_{Bn}^0$, say, for that particular value of $n$. It follows that $t_{Bn}^0$, which is a function of $n$, $I_A$, and $\omega$, can be evaluated by solving the coupled Eqs. (7) and (17) at $I_t = I_t^0 + I_A$. This is equivalent to solving (7) for the case $\omega t_{Bn} = 2n\pi + \pi$.

After some algebra, we obtain

$$t_{Bn}^0 = \tau_j \ln \frac{2I_{t}^0 + I_A (1 + \sin^2 \theta)}{I_A \cos^2 \theta},$$  \hfill (18)

supplemented by

$$\omega t_{Bn}^0 = (2n + 1)\pi.$$  \hfill (19)

These are coupled equations since $\theta$ is a function of $\omega$ [see Eq. (6)]. By solving (18) and (19) we obtain $t_{Bn}^0$ as a function of $I_A$, and the results are shown in Fig. 2. As illustrated by that figure, $t_{Bn}^0$ is a decreasing function of $I_A$, and falls very closely to the $n = \infty$ curve if $n \geq 1$. The $n = \infty$ curve [which, from (19), corresponds to $\omega \to \infty$ and hence, from (6), $\theta \to 0$] can be obtained analytically from (18) as

$$t_{Bn}^0 = t_{Bn}^0 - \infty = \tau_j \ln (1 + 2I_{t}^0/I_A),$$  \hfill (20)

which is the absolute optimum Bloch-oscillation period at the resistive steps for a fixed $I_A$. It is interesting to note that since (20) is derived under the condition

![FIG. 2. Optimum Bloch-oscillation period $t_{Bn}^0$, calculated from (18), as well as the corresponding $\omega$ values [as given by Eq. (19)], as a function of $I_A$. Only the $n = 0, 1$ and $n = \infty$ cases are given for the $t_{Bn}^0$ curves since the lines between $n = 1$ and $n = \infty$ are not identifiable in the present scale. However, for the $\omega$ curves, $n$ values from 0 to 3 are given. Also, $\tau_j$ is the characteristic time of the junction and $I_{t}^0 = e/\tau_j$. The inset illustrates $t_{B1}$ as a function of $\omega$ for a particular choice of $I_A$ and hence $t_{B1}^0$.](image-url)
\( I_n = I_t^0 + I_A \), one observes that the \( I_t^0 \) of (20) is formally the same as the \( t_B \) of (9) if one replaces the \( I_A \) in (20) by \( I_A = I_t^0 \). Also, in the weak-damping limit \( (I_t^0 \to 0) \), from (20) one sees that \( t_B^0 = 2e \omega / I_A \) similar to the corresponding result of (9) viz. \( t_B = 2e \omega / I \). This is not surprising because in the \( \omega \to \infty \) limit, the time intervals \( 2 \pi / \omega \) between the \( t_{Bn} \) of (17) become infinitesimally small and \( t_{Bn} \) approaches the continuous values \( t_B \) of (9). On the other hand, the fact that \( t_B^0 \) is reached only when \( 2\pi / \omega \to 0 \), implies that in choosing a small \( t_{Bn} \) and a large \( 2\pi / \omega \), for the purpose of optimizing the observation of the Bloch oscillations, one must make a compromise. Since from Fig. 2 one observes that the \( t_{Bn}^0 \) values are all close to the \( t_B^0 \) given by (20), it is clear that the best optimization is achieved by focusing our attention in achieving a large value for \( 2\pi / \omega \) (while noting that \( \omega \) cannot be made too small as otherwise \( t_{Bn}^0 \) would get too large) which, from (19), corresponds to a small value of \( n \). Also plotted in Fig. 2 are the values of \( \omega \) corresponding to \( t_{Bn}^0 \) for \( n = 0,1,2,3 \), calculated by (19). These curves demonstrate that in order to reach the optimum Bloch-oscillation period \( t_{Bn}^0 \), the value of \( I_A \) and \( \omega \) must be tuned to values represented by these curves. In fact, in the weak-damping limit \( (\tau_j \to \infty) \), one can deduce from (18) and (19), the relation at the various \( t_{Bn}^0 \), as

\[
\frac{I_n}{e \omega} = \frac{I_t^0 + I_A}{e \omega} = \frac{2}{(2n + 1)\pi}.
\]

Equation (21) explains why the \( \omega vs I_A \) curves in Fig. 2 are straight lines in the large \( \omega \tau_j \) region.

Physically, the above discussion is important, because if one attempts to detect the Bloch oscillations by means of ac current effects, then the smaller the \( t_{Bn} \) achieved the better chance the experiment succeeds. Our results (18)–(21) provide a way to pin-point the exact value of \( t_{Bn}^0 \), and give some hint as to why the detection of Bloch oscillations is so difficult. We note from (18) and Fig. 2 that one should take the amplitude \( I_A \) as large as possible for the purpose of achieving a small value for \( t_{Bn}^0 \). Once \( I_A \) is fixed, the frequency \( \omega \) should be tuned to a value determined by (21) using the smallest possible \( n \) value. As an example of the frequency detuning effect on the \( t_{Bn} \), in the insert of Fig. 2 we plot the \( t_{Bn} \) obtained from (7) and (17) at \( I_A/I_t^0 = 3.0 \) and \( n = 1 \) in the frequency region \( 9.1 < \omega \tau_j < 18.2 \) (the latter value of \( \omega \tau_j \) corresponds to \( t_{Bn}^0 \)). The insert shows that when \( \omega \tau_j \) deviates from its tuned value (here 18.2), \( t_{Bn} \) increases dramatically from its optimum value \( t_{Bn}^0 \), worsening the chance of observing the Bloch oscillation.

Finally, the solutions for \( I_n \) and \( t_{Bn} \) can be obtained by solving the coupled equations (7) and (17). When \( n \gg 1 \), an asymptotic form of \( I_n \) is obtained as

\[
I_n = I_t + [I_t + I_A \sin^2 \theta] \exp \left( -\frac{2n \pi}{\omega \tau_j} \right),
\]

where \( I_t^0 = e / \tau_j \) is the threshold current at \( I_A = 0 \), and \( I_t \) is given by (16). The low indexed solution \( (n = 0,1) \) for \( I_n \) and \( t_{Bn} \) will be discussed later when we study the \( I-V \) curves.

### B. I-V curves

Next, we study the \( I-V \) curve for a small Josephson junction described by the quasicharge equation (4). In general, the \( I-V \) curve is obtained by solving the coupled equations (7) and (8) at a fixed value of \( I_A \) and \( \omega \), and by selecting the smallest value of \( t_B \) which is consistent with (7), as discussed above. As an example, in Fig. 3 we plot the solutions \( (I/I_t^0 vs V/I_t^0 R_I) \) of the coupled equations of (7) and (8) at \( \omega \tau_j = \pi \) and \( I_A/I_t^0 = 0.0, 0.3, 1.0, 3.0 \). The main effects of the ac current, as shown in Fig. 3, can be summarized as follows.

(i) The value of the threshold current decreases from \( I_t^0 \) at \( I_A = 0 \) to the \( I_t \) of (16). Using (6) and (16), we conclude that when \( (I_A/I_t^0) > \sqrt{\omega \tau_j + 1} \), \( I_t \) is totally suppressed, and the ohmic part of the \( I-V \) curves disappears.

(ii) Defining \( V_n \) as the voltage corresponding to \( I_n \), one observes that horizontal resistive steps appear in the \( I-V \) curves at the values of \( I = I_n \) and \( V = V_n \), in the region \( I_t \leq I_n \leq I_t^0 + I_A \). For \( n \gg 1 \), \( I_n \) is given by (22), and \( V_n \) is evaluated by (8) at a \( t_{Bn} \) given by (17), with the result

\[
\frac{V_n}{V} = R_I \left[ I_t^0 \frac{\omega \tau_j}{\omega \tau_j^2} - \frac{I_A \omega \tau_j}{2I_t^0 \sqrt{\omega \tau_j^2 + 1}} \right],
\]

where \( I_t \) is defined by (16). Equations (22) and (23) show

![FIG. 3. Current-voltage (I-V) curves, calculated from Eq. (22), at \( \omega \tau_j = \pi \), for a dc current-biased Josephson junction with additional ac bias \( I_A \cos \omega t \) in the \( E_s \gg E_j \) limit (\( E_s \) = charging energy, \( E_j \) = Josephson critical energy), at three different \( I_A \) values, are presented. Also, \( I_t^0 \) is the threshold current, and \( C_j \) is the junction capacitance.](image-url)
that for $I$ close to $I_c$ (corresponding to $n \gg 1$) the resistive steps in the $I-V$ curves take place at a current value obeying an exponential law and at a voltage value obeying a power law, with respect to $n$. Between these steps the voltage increases almost linearly with current and the corresponding ohmic resistance is smaller than $R_J$.

(ii) Another kind of noteworthy behavior, the voltage bumps (the points $B$ in Fig. 3), which correspond to a divergent differential conductance $dI/dV$, also appear in the $I-V$ curves. A voltage bump may appear at a current value above the first resistive step, which is the case of $I_A/I_A^0 = 0.3$ and 1.0 in Fig. 3. The order is reversed for large $I_A$ (see the $I_A/I_A^0 = 3.0$ case in Fig. 3), where the first resistive step (point $A$ in Fig. 3) appears at a current value above the first voltage bump. Also, we find that the relative positions of $A$ and $B$ depend on the value of $\omega\tau_J$. This is further illustrated in Fig. 4, where we plot the $I-V$ curves for $I_A/I_A^0 = 1.0$ in the $I_A/I_A^0 \geq 1$ region at several values of $\omega\tau_J = 1.2, 2.4, \pi, 4\pi$. As can be observed from the figure, the first resistive steps, points $A$, appear at a current above (below) the first voltage bump, points $B$, for relatively small (large) $\omega\tau_J$ values.

(iv) When $I > I_A^0 + I_A$, no anomalous behavior appears, and the $I-V$ curves fall close to those of the $I_A = 0$ case, where both $V$ and $I_B$ tend to zero as $I \rightarrow \infty$.

We note that the $I-V$ curves shown in Fig. 3 for the analytic solutions (7) and (8) are considerably different from the numerical results displayed in Fig. 5 of Ref. 1. We will comment further on this below.

FIG. 4. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$.

C. Frequency dependence of resistance steps

We now turn to a discussion of the frequency ($\omega = 2\pi f$) dependence of the resistive steps in the $I-V$ curves, which is a quantity of great experimental interest. For this purpose it is convenient to write (8) in the form

$$\bar{I}_n \equiv \frac{I_n - V_n}{R_J} = \frac{2e}{\tau_B} \frac{I_A\sin\omega t_{Bn}}{\omega t_{Bn}}$$

$$= a_n 2e f - b_n, \quad n = 0, 1, 2, \ldots,$$

where

$$a_n \equiv 2\pi/\omega t_{Bn}, \quad b_n \equiv I_A\sin\omega t_{Bn}/\omega t_{Bn},$$

and $\omega t_{Bn}$ is given by (17). Equation (24) tells us that for a fixed ac current bias $I_A\cos2\pi ft$, there exists an infinite set of $I_n, V_n, T_{Bn}, n = 0, 1, 2, \ldots$, at which the $I-V$ curves have resistive steps. In addition, since (17) implies that $\omega t_{Bn}$ is only weakly dependent on frequency for fixed $n$, it follows from (24) and (25) that $\bar{I}_n$ is a quasilinear increasing function of the external frequency $f$. This feature is manifested in Fig. 5 (discussed below). Formally, (24) is very similar to a widely used formula in the literature

$$I = \begin{bmatrix} m \\ n \end{bmatrix} 2ef,$$

where $m$ and $n$ are nonzero integers. Equation (26) is ob-

FIG. 5. dc current $\bar{I} \equiv I_n - V_n/R_J$ (in units of $I_A^0$), where $I_n$ and $V_n$ are the current and voltage at the resistive steps as calculated by (7), (8), and (17), plotted as a function of the applied frequency $\omega$ (in units of $1/\tau_J$). Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 6. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 7. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 8. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 9. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 10. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 11. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$. 

FIG. 12. Current-voltage ($I-V$) curves, calculated from Eq. (22), at $I_A = I_A^0$, for a dc current-biased Josephson junction with additional ac bias $I_A\cos\omega t$, with $E_r \gg E_J$ limit ($E_r =$ charging energy, $E_J =$ Josephson critical energy) at four different values of $\omega$. Also, $\tau_J$ is the characteristic time of the junction and $I_A^0 = e/\tau_J$.
tained in Ref. 1 in the $R_J \to \infty$ limit by the argument that the ac current can phase-lock either the basic Bloch frequency $f_B$ or one of its harmonics $(n f_B)$ or subharmonics $(f_B/m)$. In our formalism (26) is directly deducible by using (7) in the $R_J \to \infty$ weak-damping limit, where $	au_J \to \infty$ and $\theta \to 0$ so that the $I_n$ contribution in (7) disappears and we obtain $I = 2e / \tau_B = 2e f_B$. Hence (26) follows if we assume the frequency-locking condition

\[
f_B = (m / n) f.
\]

In addition, if we now use these relations in (8) then we find that $V = 0$. We conclude that (26) is the special case of (7); it is obtained by assuming (a) weak damping and (b) frequency locking and, furthermore, another consequence of these assumptions is that the average voltage, given by (8), becomes zero. We note that the $I$-$V$ curve in the region $V = 0$ is smooth, in contrast to the situation where the resistive steps (points at which the differential resistance changes abruptly) occur. Unfortunately, in the literature, (26) has been used in the latter context. Also, it has been extended to the finite $R_J$ case without revision.

In the framework of (4), our solution (24) gives the correct $I_n$-$f$ relation at the resistive steps. In various special cases, (24) reduces to some simple forms. In the $\omega \to 0$ limit, one formally has $b_n \to I_A$, and (24) becomes

\[
I_n + I_A = \frac{2e}{\tau_B}
\]

where $\omega \tau_B$ is finite given by (17). One notices (27) is similar to (10) with $I$ replaced by $I_n + I_A$, which reflects that in this case there is no actual ac bias. Also, one can show that this is the case similar to the $I_n = 0$ case where the abrupt change of $I$-$V$ curves occurs only in a region close to $I = I_A$. When $\omega \tau_B \gg 1$ ($n \gg 1$), one has $q_n \approx n^{-1}$ and $b_n \approx 0$, and (24) approximately takes the form (26).

For the most important cases of (24) relevant to the experimental measurement, $\omega \tau_B$ takes the smallest possible values in (13) which correspond to $n = 0, 1$. In this case, (24) can be solved together with the coupled equations (7) and (17). We have carried out such calculations, and the results of three typical values $I_A / I^0 = 0.3, 1.0$, and 3.0, are presented in Fig. 5. For each value of $I_A$, two lines of $I_n$ (in units of $I^0_n$) vs $\omega \tau_J$ are plotted, representing, respectively, the $n = 0$ and the $n = 1$ solutions of (24). The $n = 0$ lines fall in the small $\omega \tau_J$ region and have a slope larger than $\pi^{-1}$ (the dashed line in Fig. 5), whereas the $n = 1$ lines are in the relatively large $\omega \tau_J$ region and have a slope less than $\pi^{-1}$. Each line spans a certain $\omega \tau_J$ region reflecting the fact that at fixed $n$ the $\omega \tau_B$ in (17) takes some restricted values as discussed before. The upper end of each line corresponds to the special point $I_n = I_A + I^0_n$, where the $I_B$ approaches the optimum value $I^0_B$, as related to that particular line (see also Fig. 2 and the insert there). The above discussion is based on the assumption of fixed $I_A$. If one wants to detect the resistive steps (due to the ac-bias current) with an optimized Bloch-oscillation time $\tau_B^0$ by using (19), (24), and (25) the resulting $I_n$ vs $\omega \tau_J$ relation can readily be derived as

\[
I_n = \frac{2}{2n + 1 - 2e f}.
\]

We note that (28) is a general form of (21), where one has $R_J \to \infty$. Also we note that the upper ends of the three fixed $I_A$ curves seen in Fig. 5 will fall on the $n = 0$ or $n = 1$ lines of (28), respectively.

V. CONCLUSIONS AND COMPARISON WITH EXPERIMENT

In summary, in this paper we have presented an analytic analysis of the Bloch oscillations in small-capacitance Josephson junctions in the $E_c \gg E_J$ limit. After obtaining the exact analysis for the quasicharge $q(t)$ in the absence of fluctuations, we derived expressions (7) and (8) for the current and voltage, respectively, by adopting the physical picture of Bloch oscillations in small Josephson junctions as periodic discrete transfer of Cooper pairs. In the absence of an ac current, we showed that the well-known analytic expressions (9)-(11) are recovered from the general results (7) and (8). When $I_J \neq 0$, we have studied the ac effects on the Bloch-oscillation period $\tau_B$, the $I$-$V$ curves, and the $I$-$f$ relations. Our analysis is focused on the resistive steps, where at a current $I_n$ the Bloch-oscillation time $\tau_B$, as well as the voltage $V_n$ take a sudden jump (divergent differential resistance $dV/dI$). Our main conclusions are the following.

(i) The dc current $I$ is a decreasing function of $\tau_B$ with periodic downward steps at a sequence of particular times $\tau_B^0$ (see Fig. 1), and the time interval between the two consecutive $\tau_B^0$ can be estimated by (17) as approximately $2\pi / \omega$. This anomalous $I$-$\tau_B$ behavior is the cause of the resistive steps that appear in the $I$-$V$ curves (see Fig. 3) where $dV/dI$ diverges.

(ii) At a fixed value of the amplitude $I_A$ of the ac current, the largest dc current, where a resistive step exists, is equal to $I_A^0 + I_A$. In this case, if one tunes the frequency $\omega$ to an optimum value determined by (18) and (19), then the Bloch-oscillation period at this resistive step reaches the optimum value $\tau_B^0$, having a phase-lock relation (19) with $\omega$. Also, a low $n$ value is achieved with a relatively small applied $\omega$ (see Fig. 2), which corresponds to wide resistive steps. This suggests that the $n = 0, 1$ resistive steps and the related physical quantities are of prime relevance to the detection of the Bloch oscillations.

(iii) The position of the resistive steps on the $I$-$V$ curves at a fixed $I_A$, as described by (24) and (25), has a quasilinear dependence on $f$ as is illustrated in Fig. 5. However, if the values of $I_A$ and $\omega$ are tuned according to (18) and (19) so that an optimum value $\tau_B^0$ for the system is reached, then (24) reduces to the simple linear form (28).

Experimentally, there is a recent study of the low-temperature behavior of single ultrasmall Josephson junctions screened from a low-impedance environment by special high-resistance resistors. Under irradiation with microwave frequencies, $\omega = 3.5$ to $10$ GHz, the values of the dc current, at the point where the $dV/dI$ displays a peak, were recorded. This led the authors of Ref. 4 to the suggestion that this is due to the...
FIG. 6. Comparison between the theory (full curves) and the experimental data (points, taken from Ref. 4), on the dc current $I$ (at the microwave-induced peaks of $dV/dI$) as a function of the applied frequency $f$. The dashed line corresponds to the relation $I = 2ef$.

periodic electrical recharging of the junction by discrete Cooper pairs, and to an explanation based on $I = 2ef$, a result which follows from the "orthodox" theory of Bloch oscillations. On the other hand, the studies presented here demonstrate that the relation $I = 2ef$ is not directly related to the resistive steps and is only correct in the weak-damping limit, where $R_J \to \infty$. We have shown that the actual $I$-$f$ relation responsible for the resistive steps is given by (24) instead of $I = 2ef$. Also, from the data presented in Ref. 4, one has $R_J \approx 200$ kΩ and $C_J \approx 4.8 \times 10^{-16}$ F, which corresponds to a finite $\tau_J \approx 9.6 \times 10^{-11}$ s (but the absolute value of $I_A$ is not available). Therefore, it is of great interest to see what will happen if we compare the general formula (24) with the experimental results of Ref. 4 at its actual value of $R_J$ and $\tau_J$. For this purpose, in Fig. 6 we plot the dc current $I$ at the $n = 1$ resistive steps as calculated by (24) [supplemented by (7) and (17)], as a function of $f$ in the range of 2–10 GHz. Three values of ac amplitudes $I_A = 0.3, 0.8, 1.6$ (nA) are used in the calculations. As can be seen from the figure, the experimental data in the $f \sim 4.0$ GHz range falls close to the theoretical curves, while the data in the $f \sim 10$ GHz range is a little above the theoretical curves. Given the fact that the values quoted in Ref. 4 for the $R_J$ and $C_J$ used in our calculation are only estimates and also noting that the junctions used in the experiment do not fulfill the $E_c \gg E_J$ criterion (where the present theory applies), what we can conclude here is that both the experimental data of Ref. 4 and our theoretical results provide very useful information for the resistive steps on the $I$-$V$ curves and much more work needs to be done toward a full understanding of Bloch oscillations in small Josephson junctions.

Finally, we mention that in a recent paper by Nazarov and Odintsov, it is argued that one should be careful with a Bloch-oscillation interpretation for the experimental observations of Ref. 4. These authors pointed out that the same observations can be interpreted in a completely different way as an "incoherent" photon-assisted macroscopic tunneling of the Josephson phase.

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