Relativistic quadrupole moment

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The relativistic quadrupole-moment contribution to the spin precession (and orbit precession) of a binary system consisting of two rotating bodies with arbitrary masses and quadrupole moments is examined.

I. INTRODUCTION

The relativistic spin precession of either body in a binary system (as well as the relativistic orbit precession) was derived by Barker and O’Connell using quantum-theoretical techniques. The spin precession of body 1 was found to be

$$\tilde{\Omega}_{\text{as}}^{(1)} = \tilde{\Omega}_{\text{as}}^{(1)} \times \tilde{S}^{(1)},$$

with

$$\tilde{S}^{(1)} = \tilde{S}_{\text{as}}^{(1)} + \tilde{S}_{\text{sr}}^{(1)} + \tilde{S}_{\text{ar}}^{(1)},$$

while the orbit precession was found to be

$$\tilde{L}_{\text{as}} = \tilde{\Omega}_{\text{as}} \times \tilde{L}, \quad \tilde{A}_{\text{as}} = \tilde{\Omega}_{\text{as}} \times \tilde{A},$$

where $$\tilde{S}^{(1)}$$ is the spin of body 1, $$\tilde{L}$$ is the orbital angular momentum, and $$\tilde{A}$$ is the Runge-Lenz vector. The quadrupole moment of body 1 gives rise to the terms $$\tilde{Q}_{\text{as}}^{(1)}$$ and $$\tilde{Q}_{\text{sr}}^{(1)}$$ [see Eqs. (5) and (6)] which are the terms of interest for this paper.

II. QUADRUPOLE-MOMENT EFFECTS

The quadrupole-moment contributions (due to body 1) given by Barker and O’Connell are

$$\tilde{Q}_{\text{as}}^{(1)} = \frac{Gm_2^2\Delta J^{(1)}}{2S^{(1)}\mu^2(1-e^2)^{7/2}} \times [\tilde{S}^{(1)} - 3(\tilde{\Omega} \cdot \tilde{S}^{(1)})\tilde{\Omega}],$$

$$\tilde{Q}_{\text{sr}}^{(1)} = -\frac{3Gm_2\Delta J^{(1)}}{4La^2(1-e^2)^{3/2}} \times [2(\tilde{\Omega} \cdot \tilde{S}^{(1)})\tilde{S}^{(1)} + \{1 - 5(\tilde{\Omega} \cdot \tilde{S}^{(1)})^2\} \tilde{\Omega}],$$

where $$a$$ is the semimajor axis, $$e$$ is the eccentricity, $$\tilde{\Omega}$$ is a unit vector in the $$\tilde{L}$$ direction, $$\tilde{S}^{(1)}$$ is a unit vector in the $$\tilde{S}^{(1)}$$ direction, and $$m_2$$ is the mass of body 2. The quadrupole moment $$\Delta J^{(1)}$$ of body 1 is equal to $$2\Delta J^{(1)}$$ where $$\Delta J^{(1)}$$ is the moment of inertia about the polar axis minus the moment of inertia about an equatorial axis. The present paper is to directly calculate a relativistic replacement for Eq. (7) which can then be used in Eqs. (5) and (6) to obtain final results that can be compared to the corresponding results of McCrea and O’Brian for general relativity in the weak-field limit Poisson’s equation for the gravitational potential $$\phi$$ is $$\Delta \phi = 4\pi G\rho$$, where the effective mass density is given by (note $$\Delta \phi = 4\pi G\rho$$ for Newtonian theory)

$$\rho_e = \frac{(T_{\text{co}} + T_{\text{mu}})/c^2}{},$$

and $$T_{\text{mu}}$$ is the matter energy-momentum tensor. We are using a notation where $$x_i = (x_i, x_i)$$ and $$x_q = i x_i = i c t$$. We shall neglect all gravitational
binding effects and assume that in the absence of rotation we have a sphere of uniform density held together by elastic forces. Thus, the special-relativity version of Eq. (8) was given where \( \sigma_{\mu\nu} = \delta_{\mu\nu} \). The relativistic replacement for Eq. (7) is thus (since in Poisson’s equation \( \rho_\nu \) is replaced by \( \rho_\nu \))

\[
\Delta I^{(1)} = \frac{1}{2} \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r}) .
\]

(9)

The energy-momentum tensor for the elastic continua is given by

\[
T_{\mu\nu} = \rho \frac{d}{d\tau} U_{\mu} U_{\nu} + S_{\mu\nu} ,
\]

where \( U_{\mu} = dx_{\mu}/d\tau \) and

\[
S_{ik} = t_{ik} - t_{ii} U_i U_k / c^2 ,
\]

\[
S_{44} = - t_{i4} U_i U_4 / c^2 .
\]

(11)

The rest mass of a rotating body is

\[
m_0 = \int \rho \, dV = \int \rho \, dV ,
\]

(12)

where

\[
\rho_\ast = \rho_0 (1 - \beta^2/c^2)^{1/2} ,
\]

\[
dV = dV_0 (1 - \beta^2/c^2)^{1/2} .
\]

(13)

Thus, the conserved mass density \( \rho_\ast \) for a spherically symmetric (in \( \rho_\ast \)) rigid body does not depend on its angular velocity (in a system where the center of mass is at rest). It is easy to see that

\[
T_{00} = \rho_\ast \rho_0 (1 - \beta^2/c^2) + O(c^{-2}) ,
\]

(14)

\[
T_{ii} = \rho_\ast \rho_0 (1 + \beta^2/c^2) + O(c^{-2}) .
\]

(15)

We need to calculate \( t_{ii} \) to first order only (i.e., the Newtonian approximation). We find that (see Appendix)

\[
t_{ii} = - (1 + \frac{\gamma}{7 + 5\alpha}) \rho_\ast \rho_0 + \frac{3}{2} \frac{1 + \gamma}{7 + 5\alpha} \rho_\ast (\overline{r}) + O(c^{-2}) ,
\]

(16)

where \( \alpha \) is Poisson’s ratio and \( \rho_\ast (\overline{r}) \) is a spherically symmetric term which cannot contribute to the quadrupole moment. It then follows that

\[
\rho_\ast = \rho_0 + \frac{3}{2} \frac{1 + \gamma}{7 + 5\alpha} \rho_\ast (\overline{r}) + O(c^{-2}) .
\]

(17)

Using the result of Eq. (17) in Eq. (9) we obtain

\[
\Delta I^{(1)} = \Delta I_R^{(1)} + \Delta I_{\ast}^{(1)} ,
\]

(18)

where

\[
\Delta I_R^{(1)} = \frac{1}{2} \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r})
\]

\[
= \frac{1}{2} \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r}) ,
\]

(19)

is the Newtonian contribution and \( \overline{r} \) denotes the unstrained position from the center of mass and \( \int dV \) is over the sphere of body 1. The result \( \Delta I_R^{(1)} \) for uniform \( \rho_\ast (\overline{r}) \) was found to be

\[
\Delta I_R^{(1)} = \frac{5}{14} \left\{ \frac{(1 + \alpha)(13 + 9\alpha)}{7 + 5\alpha} \right\} \frac{(r^4 \omega^{(1)})}{Y_V} ,
\]

(20)

where \( Y \) is Young’s modulus and \( V_1 \) is the volume of body 1 (in the system where the center of mass is at rest). The relativistic contribution \( \Delta I_{\ast}^{(1)} \) for uniform \( \rho_\ast (\overline{r}) \) is

\[
\Delta I_{\ast}^{(1)} = \frac{3}{2} \frac{1 + \gamma}{7 + 5\alpha} \frac{(r^4 \omega^{(1)})}{m_0 c^2} \times \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r}) ,
\]

(21)

and to the order of approximation being considered we need integrate only over the sphere using the unstrained distances. We can put Eq. (21) in the form

\[
\Delta I_{\ast}^{(1)} = \frac{5}{14} \left\{ \frac{(1 + \alpha)(13 + 9\alpha)}{7 + 5\alpha} \right\} \frac{(r^4 \omega^{(1)})}{m_0 c^2} ,
\]

(22)

or

\[
\Delta I_{\ast}^{(1)} = \frac{3}{4} \frac{1 + \gamma}{7 + 5\alpha} (r^4 \omega^{(1)}) / c^2 ,
\]

(23)

where we define

\[
I^{(1)} = \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r}) = \frac{3}{2} \int \rho \, r^2 dV = \frac{3}{2} J ,
\]

(24)

\[
J^{(1)} = \int dV (r^2 - 3z^2) \rho_{\ast} (\overline{r}) = \frac{1}{15} \int \rho \, r^4 dV = \frac{1}{15} J ,
\]

(25)

Body 1 rotates with an angular velocity \( \omega^{(1)} \) and it should be noted that \( I^{(1)} = I^{(1)}_R \) plus higher-order terms.

For a numerical example involving a real physical object (of uniform mass density \( \rho_\ast \) and held together by elastic forces) we shall choose the gyro\(^2\) of the relativity gyroscope experiment. This fused quartz gyro has a rest radius \( r_0 = 2 \) cm and will spin at \( \omega = 400 \) rad/sec. We also have \( \rho_\ast = 2.2 \) g/cm\(^3\), \( Y = 7 \times 10^{11} \) dyn/cm\(^2\), and \( \gamma = 0.16 \) for fused quartz at \( 2^\circ \)K. Noting that \( I = 11 \pi r_0^2 \rho_\ast = 118 \) g cm\(^2\) along with Eqs. (20) and (22), we find that \( \Delta I_{\ast}/I = 6.09 \times 10^{-10} \) and \( \Delta I_{\ast}/I = 1.36 \times 10^{-12} \).

McCrea and O’Brien used a different physical model for their rotating spheres (with spherically symmetric mass density \( \rho \)) give final results that are the same as one would obtain by using

\[
\Delta I_{\ast}^{(1)} = 2 (\omega^{(1)}) J A / c^2
\]

(26)

in Eqs. (5) and (6). It should also be noted that their \( J A \) and \( J A \) are defined by Eqs. (24) and (25) with a spherically symmetric \( \rho = T_{00}/c^2 \). From Eqs. (8) and (15) we find that
\[ \rho_e = \rho + \rho v^2/c^2 + t_{11}/c^2 + O(c^{-4}). \]  
\[ (27) \]

Noting that the \( t_{11} \) used by McCrea and O'Brien is spherically symmetric the result of Eq. (26) follows from Eq. (27).

Finally let us note that the quadrupole moment of a black hole is a relativistic quadrupole moment. If body 1 is a black hole one should use

\[ \Delta J_{BH}^{(1)} = (\Delta^1)^2 / m_1 c^2 \]  
\[ (28) \]
in Eqs. (5) and (6).

### III. Conclusion

We assert that relativistic expressions for \( \Delta J^{(1)} \) such as Eqs. (9), (26), and (28) can be correctly used in Eqs. (5) and (6) which were first derived using nonrelativistic physics. We shall assume that the two bodies are at sufficient distance so that post-Newtonian approximations hold. We shall give three arguments for the above assertion.

(i) The effect of \( \Delta J^{(1)} \) on body 2 should depend on its magnitude and not on how the magnitude of \( \Delta J^{(1)} \) splits between Newtonian and relativistic parts since the gravitational potential \( \phi \) produced in part by \( \Delta J^{(1)} \) and acting on body 2 does not depend on that split. Thus, the corresponding equal and opposite reaction of body 2 (due to \( \Delta J^{(1)} \)) on body 1 should depend only on the magnitude of \( \Delta J^{(1)} \) and not on the nature of the split.

(ii) Using the methods of this paper and the McCrea–O’Brien model for rotating spheres we obtained \( \Delta J^{(1)} \) of Eq. (26) which when used in Eqs. (5) and (6) gave the same results as directly obtained by McCrea and O’Brien using relativistic physics.

(iii) Using the black-hole result \( \Delta J_{BH}^{(1)} \) of Eq. (28) in Eqs. (5) and (6) gives the same results as directly obtained by D’Eath using relativistic physics.

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### Appendix

Let us consider a body at rest to be a perfect sphere of radius \( r_0 \) with mass \( m_0 \) and uniform mass density \( \rho_0 \). We shall denote the coordinates of the point occupied by a particle in the unstrained state of the body by \( \mathbf{r} \) and the coordinates of the point occupied by the same particle in the strained state by \( \mathbf{r}' = \mathbf{r} + \mathbf{u} \). If the body rotates with an angular velocity \( \omega \) about the z axis the complete expressions for the displacements are given by

\[ u_i = \frac{1}{2} \rho_0 \omega^2 (A r_0^2 + B x_j^2 + C') x_i, \]  
\[ (A1) \]

\[ u_3 = \frac{\rho_0 \omega^2}{r_0} \]  
\[ (A2) \]

where capital indices (\( I, J, K \), etc.) take on values of 1 and 2, and

\[ A = \frac{1}{5(\lambda + 2\mu)} \left( \frac{5\lambda + 6\mu}{3\lambda + 2\mu} \right) + \frac{(4\lambda + 3\mu)}{\mu(19\lambda + 14\mu)}, \]  
\[ (A3) \]

\[ B = \frac{-1}{5(\lambda + 2\mu)} - \frac{4(\lambda + 2\mu)}{\mu(19\lambda + 14\mu)}, \]  
\[ (A4) \]

\[ C = \frac{-1}{5(\lambda + 2\mu)} - \frac{4(\lambda + 2\mu)}{\mu(19\lambda + 14\mu)}, \]  
\[ (A5) \]

\[ \alpha = \frac{1}{5(\lambda + 2\mu)} \left( \frac{5\lambda + 6\mu}{3\lambda + 2\mu} \right) - \frac{(8\lambda + 6\mu)}{\mu(19\lambda + 14\mu)}, \]  
\[ (A6) \]

\[ \beta = \frac{-1}{5(\lambda + 2\mu)} \frac{(6\lambda + 5\mu)}{\mu(19\lambda + 14\mu)}, \]  
\[ (A7) \]

\[ \gamma = \frac{-1}{5(\lambda + 2\mu)} \frac{(3\lambda + 2\mu)}{\mu(19\lambda + 14\mu)}. \]  
\[ (A8) \]

We also have

\[ \lambda = \frac{Y_0}{(1 + \sigma)(1 - 2\sigma)}, \quad \mu = \frac{Y_0}{2(1 + \sigma)}. \]  
\[ (A9) \]

The strain tensor \( e_{ij} \) is given by

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right), \]  
\[ (A10) \]

and the stress tensor \( \sigma_{ij} \) (note \( \sigma_{ij} = -t_{ij} \)) is given by

\[ \sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}, \]  
\[ (A11) \]

and hence

\[ t_{ij} = -\sigma_{ij} = -(3\lambda + 2\mu) e_{ij}. \]  
\[ (A12) \]

Evaluating Eq. (A12) we find that

\[ t_{ij} = -\frac{1}{2} \rho_0 \omega^2 (3\lambda + 2\mu) [(4B + \beta - 2C - 3\gamma)x_i^2] + t_{ij}^{(SS)} \]

\[ = -\left( \frac{1 + \sigma}{7 + 5\sigma} \right) \rho_0 \omega^2 x_i^2 + t_{ij}^{(SS)}, \]  
\[ (A13) \]

where

\[ t_{ij}^{(SS)} = -\frac{1}{2} \rho_0 \omega^2 (3\lambda + 2\mu) [(2A + \alpha)x_i^2] + (2C + 3\gamma)x_i^2. \]  
\[ (A14) \]

The force density \( k_i \) is given by

\[ k_i = \rho_0 \omega^2 a_i \]  
\[ (A15) \]

where \( a \) is the acceleration. Evaluating Eq. (A15) we obtain the result

\[ a_i = -\omega^2 x_i, \quad a_3 = 0. \]  
\[ (A16) \]
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10For spherically symmetric $\rho_0$ we have $\int \rho_0 r^2 dV = 3 \int r r^2 dV = 5 \int r^2 dV$.
11See second paper of Ref. 1.