Lecture 21: THU 01 APR 2010
Ch. 31.4–7: Electrical Oscillations, LC Circuits, Alternating Current
LC Circuit: At $t=0$ 1/3 Of Energy $U_{\text{total}}$ is on Capacitor $C$ and Two Thirds On Inductor $L$. Find Everything! (Phase $\varphi_0$=?)

\[
U_B(t) = \frac{1}{2} L \left[ q_0 \omega \sin(\omega t + \varphi_0) \right]^2
\]

\[
U_E(t) = \frac{1}{2C} \left[ q_0 \cos(\omega t + \varphi_0) \right]^2
\]

\[
U_B(0) = \frac{1}{2} L \left[ q_0 \omega \sin(\varphi_0) \right]^2 = \frac{U_{\text{total}}}{3}
\]

\[
U_E(0) = \frac{1}{2C} \left[ q_0 \cos(\varphi_0) \right]^2 = \frac{2U_{\text{total}}}{3}
\]

\[
\frac{U_B(0)}{U_E(0)} = \frac{\frac{1}{2} L \left[ q_0 \omega \sin(\varphi_0) \right]^2}{\frac{1}{2C} \left[ q_0 \cos(\varphi_0) \right]^2} = \frac{U_{\text{total}}}{2U_{\text{total}} / 3} = \frac{1}{2}
\]

\[
\omega = \frac{1}{\sqrt{LC}}
\]

\[
q_0 = VC
\]

\[
\tan(\varphi_0) = \frac{1}{\sqrt{2}}
\]

\[
\varphi_0 = \arctan\left( \frac{1}{\sqrt{2}} \right) = 35.3^\circ
\]

\[
q = q_0 \cos(\omega t + \varphi_0)
\]

\[
i(t) = -q_0 \omega \sin(\omega t + \varphi_0)
\]

\[
i'(t) = -\omega^2 q_0 \cos(\omega t + \varphi_0)
\]

\[
V_L(t) = -\frac{q_0}{C} \cos(\omega t + \varphi_0)
\]

\[
V_C(t) = \frac{q_0}{C} \cos(\omega t + \varphi_0)
\]
Damped LCR Oscillator

Ideal LC circuit without resistance: oscillations go on forever; $\omega = (LC)^{-1/2}$

Real circuit has resistance, dissipates energy: oscillations die out, or are “damped”

Math is complicated! Important points:

- Frequency of oscillator shifts away from $\omega = (LC)^{-1/2}$
- Peak CHARGE decays with time constant = $\tau_{QLCR} = 2L/R$
- For small damping, peak ENERGY decays with time constant $\tau_{ULCR} = L/R$

$$U_{max} = \frac{Q^2}{2C} e^{-\frac{Rt}{L}}$$
Damped Oscillations in an RCL Circuit

If we add a resistor in an RL circuit (see figure) we must modify the energy equation, because now energy is being dissipated on the resistor: \[ \frac{dU}{dt} = -i^2 R. \]

\[ U = U_E + U_B = \frac{q^2}{2C} + \frac{Li^2}{2} \rightarrow \frac{dU}{dt} = \frac{q}{C} \frac{dq}{dt} + Li \frac{di}{dt} = -i^2 R \]

\[ i = \frac{dq}{dt} \rightarrow \frac{di}{dt} = \frac{d^2 q}{dt^2} \rightarrow L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0. \] This is the same equation as that of the damped harmonics oscillator: \[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0, \] which has the solution:

\[ x(t) = x_m e^{-bt/2m} \cos (\omega' t + \phi). \] The angular frequency \[ \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \]

For the damped RCL circuit the solution is:

\[ q(t) = Qe^{-Rt/2L} \cos (\omega' t + \phi). \] The angular frequency \[ \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \]

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The equations above describe a harmonic oscillator with an exponentially decaying amplitude $Qe^{-Rt/2L}$. The angular frequency of the damped oscillator

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

is always smaller than the angular frequency $\omega = \sqrt{\frac{1}{LC}}$ of the undamped oscillator. If the term $\frac{R^2}{4L^2} \ll \frac{1}{LC}$ we can use the approximation $\omega' \approx \omega$.

$$\tau_{RC} = RC \quad \tau_{RL} = \frac{L}{R} \quad \tau_{RCL} = \frac{2L}{R}$$
Summary

• Capacitor and inductor combination produces an electrical oscillator, natural frequency of oscillator is $\omega = 1/\sqrt{LC}$

• Total energy in circuit is conserved: switches between capacitor (electric field) and inductor (magnetic field).

• If a resistor is included in the circuit, the total energy decays (is dissipated by R).
Alternating Current:

To keep oscillations going we need to drive the circuit with an external emf that produces a current that goes back and forth.

Notice that there are two frequencies involved: one at which the circuit would oscillate “naturally”. The other is the frequency at which we drive the oscillation.

However, the “natural” oscillation usually dies off quickly (exponentially) with time. Therefore in the long run, circuits actually oscillate with the frequency at which they are driven. (All this is true for the gentleman trying to make the lady swing back and forth in the picture too).
Alternating Current:

We have studied that a loop of wire, spinning in a constant magnetic field will have an induced emf that oscillates with time,

\[ \mathcal{E} = \mathcal{E}_m \sin(\omega_d t) \]

That is, it is an AC generator.

AC’s are very easy to generate, they are also easy to amplify and decrease in voltage. This in turn makes them easy to send in distribution grids like the ones that power our homes.

Because the interplay of AC and oscillating circuits can be quite complex, we will start by steps, studying how currents and voltages respond in various simple circuits to AC’s.
AC Driven Circuits:

1) A Resistor:

\[ emf - v_R = 0 \]

\[ v_R = emf = E_m \sin(\omega_d t) \]

\[ i_R = \frac{v_R}{R} = \frac{E_m}{R} \sin(\omega_d t) \]

Resistors behave in AC very much as in DC, current and voltage are proportional (as functions of time in the case of AC), that is, they are “in phase”.

For time dependent periodic situations it is useful to represent magnitudes using Steinmetz “phasors”. These are vectors that rotate at a frequency \( \omega_d \), their magnitude is equal to the amplitude of the quantity in question and their projection on the vertical axis represents the instantaneous value of the quantity.
2) Capacitors:

\[ v_C = emf = \mathcal{E}_m \sin(\omega_d t) \]
\[ q_C = C \, emf = C \mathcal{E}_m \sin(\omega_d t) \]
\[ i_C = \frac{dq_C}{dt} = \omega_d C \mathcal{E}_m \cos(\omega_d t) \]
\[ i_C = \omega_d C \mathcal{E}_m \sin(\omega_d t + 90^\circ) \]

i_C = \frac{\mathcal{E}_m}{X} \sin(\omega_d t + 90^\circ)

where \( X = \frac{1}{\omega_d C} \) "reactance"

\[ i_m = \frac{\mathcal{E}_m}{X} \]

looks like \( i = \frac{V}{R} \)

Capacitors "oppose a resistance" to AC (reactance) of frequency-dependent magnitude \( 1/\omega_d C \)

(this idea is true only for maximum amplitudes, the instantaneous story is more complex).
AC Driven Circuits:

3) Inductors:

\[ v_L = emf = \mathcal{E}_m \sin(\omega_d t) \]

\[ v_L = L \frac{di_L}{dt} \implies i_L = \frac{\int v_L \, dt}{L} \]

\[ i_L = -\frac{\mathcal{E}_m}{L\omega_d} \cos(\omega_d t) = \frac{\mathcal{E}_m}{L\omega_d} \sin(\omega_d t - 90^0) \]

\[ i_L = \frac{\mathcal{E}_m}{X} \sin(\omega_d t - 90^0) \]

\[ i_m = \frac{\mathcal{E}_m}{X} \quad \text{where} \quad X = L\omega_d \]

Inductors “oppose a resistance” to AC (reactance) of frequency-dependent magnitude \( \omega_d L \) (this idea is true only for maximum amplitudes, the instantaneous story is more complex).
Energy Transmission Requirements

The resistance of the power line $R = \frac{\rho \ell}{A}$. $R$ is fixed (220 $\Omega$ in our example).

Heating of power lines $P_{\text{heat}} = I_{\text{rms}}^2 R$. This parameter is also fixed (55 MW in our example).

Power transmitted $P_{\text{trans}} = \mathcal{E}_{\text{rms}} I_{\text{rms}}$ (368 MW in our example).

In our example $P_{\text{heat}}$ is almost 15% of $P_{\text{trans}}$ and is acceptable.

To keep $P_{\text{heat}}$ we must keep $I_{\text{rms}}$ as low as possible. The only way to accomplish this is by increasing $\mathcal{E}_{\text{rms}}$. In our example $\mathcal{E}_{\text{rms}} = 735$ kV. To do that we need a device that can change the amplitude of any ac voltage (either increase or decrease).

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The DC vs. AC Current Wars

Thomas Edison pushed for the development of a DC power network.

George Westinghouse backed Tesla’s development of an AC power network.

Nikola Tesla was instrumental in developing AC networks.

Edison was a brute-force experimenter, but was no mathematician. AC cannot be properly understood or exploited without a substantial understanding of mathematics and mathematical physics, which Tesla possessed.
The most common example is the Tesla three-phase power system used for industrial applications and for power transmission. The most obvious advantage of three phase power transmission using three wires, as compared to single phase power transmission over two wires, is that the power transmitted in the three phase system is the voltage multiplied by the current in each wire times the square root of three (approximately 1.73). The power transmitted by the single phase system is simply the voltage multiplied by the current. Thus the three phase system transmits 73% more power but uses only 50% more wire.
Against General Electric and Edison’s proposal, Westinghouse, using Tesla's AC system, won the international Niagara Falls Commission contract. Tesla’s three-phase AC transmission became the World’s power-grid standard.

Transforming DC power from one voltage to another was difficult and expensive due to the need for a large spinning rotary converter or motor-generator set, whereas with AC the voltage changes can be done with simple and efficient transformer coils that have no moving parts and require no maintenance. This was the key to the success of the AC system. Modern transmission grids regularly use AC voltages up to 765,000 volts.
The transformer is a device that can change the voltage amplitude of any ac signal. It consists of two coils with a different number of turns wound around a common iron core.

The coil on which we apply the voltage to be changed is called the "primary" and it has \( N_p \) turns. The transformer output appears on the second coil, which is known as the "secondary" and has \( N_s \) turns. The role of the iron core is to ensure that the magnetic field lines from one coil also pass through the second. We assume that if voltage \( V_p \) is applied across the primary then a voltage \( V_s \) appears on the secondary coil. We also assume that the magnetic field through both coils is equal to \( B \) and that the iron core has cross-sectional area \( A \). The magnetic flux through the primary

\[
\Phi_p = N_p BA \rightarrow V_p = -\frac{d\Phi_p}{dt} = -N_p A \frac{dB}{dt} \quad \text{(eq. 1)}
\]

The flux through the secondary

\[
\Phi_s = N_s BA \rightarrow V_s = -\frac{d\Phi_s}{dt} = -N_s A \frac{dB}{dt} \quad \text{(eq. 2)}
\]
\[ \frac{V_S}{N_S} = \frac{V_P}{N_P} \]

\[ \Phi_P = N_P BA \rightarrow V_P = -\frac{d\Phi_P}{dt} = -N_P A \frac{dB}{dt} \quad (\text{eq. 1}) \]

\[ \Phi_S = N_S BA \rightarrow V_S = -\frac{d\Phi_S}{dt} = -N_S A \frac{dB}{dt} \quad (\text{eq. 2}) \]

If we divide equation 2 by equation 1 we get:

\[ \frac{V_S}{V_P} = \frac{-N_S A \frac{dB}{dt}}{-N_P A \frac{dB}{dt}} = \frac{N_S}{N_P} \rightarrow \frac{V_S}{N_S} = \frac{V_P}{N_P}. \]

The voltage on the secondary \( V_S = V_P \frac{N_S}{N_P}. \)

If \( N_S > N_P \rightarrow \frac{N_S}{N_P} > 1 \rightarrow V_S > V_P, \) we have what is known as a "step up" transformer.

If \( N_S < N_P \rightarrow \frac{N_S}{N_P} < 1 \rightarrow V_S < V_P, \) we have what is known as a "step down" transformer.

Both types of transformers are used in the transport of electric power over large distances.  

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If we close switch S in the figure we have in addition to the primary current $I_p$ a current $I_s$ in the secondary coil. We assume that the transformer is "ideal," i.e., it suffers no losses due to heating. Then we have: $V_p I_p = V_s I_s$ (eq. 2).

If we divide eq. 2 with eq. 1 we get: $\frac{V_p I_p}{V_p N_s} = \frac{V_s I_s}{V_s N_p} \rightarrow I_P N_P = I_S N_S$.

$I_S = \frac{N_P}{N_S} I_P$

In a step-up transformer ($N_S > N_P$) we have that $I_S < I_P$.

In a step-down transformer ($N_S < N_P$) we have that $I_S > I_P$.

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