Lecture 29
Ch. 36: Diffraction
Things You Should Learn from This Lecture

1. When light passes through a small slit, it spreads out and produces a diffraction pattern, showing a principal peak with subsidiary maxima and minima of decreasing intensity. The primary diffraction maximum is twice as wide as the secondary maxima.

2. We can use Huygens' Principle to find the positions of the diffraction minima by subdividing the aperture, giving $\theta_{\text{min}} = \pm p \lambda/a$, $p = 1, 2, 3, \ldots$.

3. Calculating the complete diffraction pattern takes more algebra, and gives $I_\theta = I_0 [\sin(\alpha)/\alpha]^2$, where $\alpha = \pi a \sin(\theta)/\lambda$.

4. To predict the interference pattern of a multi-slit system, we must combine interference and diffraction effects.
Single Slit Diffraction

When light goes through a narrow slit, it spreads out to form a **diffraction pattern**.
Analyzing Single Slit Diffraction

For an open slit of width $a$, subdivide the opening into segments and imagine a Hyugen wavelet originating from the center of each segment. The wavelets going forward ($\theta=0$) all travel the same distance to the screen and interfere constructively to produce the central maximum.

Now consider the wavelets going at an angle such that $\lambda = a \sin \theta \equiv a \theta$. The wavelet pair (1, 2) has a path length difference $\Delta r_{12} = \lambda/2$, and therefore will cancel. The same is true of wavelet pairs (3,4), (5,6), etc. Moreover, if the aperture is divided into $p$ sub-parts, this procedure can be applied to each sub-part. This procedure locates all of the dark fringes.

$$\frac{\lambda}{a} = \sin \theta_p \approx \theta_p ; \quad p = 1, 2, 3, \cdots \quad \text{(angle of the $p^{th}$ dark fringe)}$$
Conditions for Diffraction Minima

\[ p \frac{\lambda}{a} = \sin \theta_p \equiv \theta_p; \quad p = 1, 2, 3, \cdots \]

(angle of the p\textsuperscript{th} dark fringe)
Pairing and Interference

Can the same technique be used to find the maxima, by choosing pairs of wavelets with path lengths that differ by $\lambda$?

No. Pair-wise destructive interference works, but pair-wise constructive interference does not necessarily lead to maximum constructive interference. Below is an example demonstrating this.

(a)  

Each pair of vectors interferes destructively. The vector sum of all six vectors is zero.

(b)  

Each pair of vectors interferes constructively. Even so, the vector sum of all six vectors is zero.
Calculating the Diffraction Pattern

We can represent the light through the aperture as a chain of phasors that “bends” and “curls” as the phase $\Delta \beta$ between adjacent phasors increases. $\beta$ is the angle between the first and the last phasor.
Calculating the Diffraction Pattern (2)

\[ E_\theta = 2r \sin \left( \frac{\beta}{2} \right) \]

\[ \beta = \frac{E_{\text{max}}}{r}; \quad r = \frac{E_{\text{max}}}{\beta} \]

\[ E_\theta = \frac{E_{\text{max}}}{\beta/2} \sin \left( \frac{\beta}{2} \right) = E_{\text{max}} \frac{\sin \alpha}{\alpha} \]

\[ \alpha = \beta = \frac{\pi a}{\lambda} \sin \theta \]

\[ I = CE^2 \]

\[ I_\theta = I_{\text{max}} \left( \frac{\sin \alpha}{\alpha} \right)^2 \]

Minima: \( \alpha = \pm m\pi \) or \( a \sin \theta = \pm m\lambda \)
The wider the slit opening $a$, or the smaller the wavelength $\lambda$, the narrower the diffraction pattern.
Radar: The Smaller The Wavelength the Better The Targeting Resolution

- **X-band:** $\lambda = 100\text{m}$
- **K-band:** $\lambda = 10\text{m}$
- **Ka-band:** $\lambda = 1\text{m}$
- **Laser:** $\lambda = 1\text{ micron}$

Outgoing waves

Reflected waves (higher frequency)
Angles of the Secondary Maxima

The diffraction minima are precisely at the angles where 
\[ \sin \theta = p \frac{\lambda}{a} \text{ and } \alpha = p\pi \] (so that \( \sin \alpha = 0 \)).

However, the diffraction maxima are not quite at the angles where 
\[ \sin \theta = (p + \frac{1}{2}) \frac{\lambda}{a} \text{ and } \alpha = (p + \frac{1}{2})\pi \] (so that \( |\sin \alpha| = 1 \)).

To find the maxima, one must look near \( \sin \theta = (p + \frac{1}{2}) \frac{\lambda}{a} \), for places where the slope of the diffraction pattern goes to zero, i.e., where 
\[ \frac{d[(\sin \alpha/\alpha)^2]}{d\theta} = 0. \] This is a transcendental equation that must be solved numerically. The table gives the \( \theta_{\text{Max}} \) solutions. Note that \( \theta_{\text{Max}} < (p + \frac{1}{2}) \frac{\lambda}{a} \).
Example: Diffraction of a laser through a slit

Light from a helium-neon laser (λ = 633 nm) passes through a narrow slit and is seen on a screen 2.0 m behind the slit. The first minimum of the diffraction pattern is observed to be located 1.2 cm from the central maximum.

How wide is the slit?

\[ \theta_1 = \frac{y_1}{L} = \frac{0.012 \text{ m}}{2.00 \text{ m}} = 0.0060 \text{ rad} \]

\[ a = \frac{\lambda}{\sin \theta_1} \approx \frac{\lambda}{\theta_1} = \frac{6.33 \times 10^{-7} \text{ m}}{6.00 \times 10^{-3} \text{ rad}} = 1.06 \times 10^{-4} \text{ m} = 0.106 \text{ mm} \]
Width of a Single-Slit Diffraction Pattern

\[ y_p = \frac{p\lambda L}{a}; \quad p = 1, 2, 3, \cdots \quad \text{(positions of dark fringes)} \]

\[ w = \frac{2\lambda L}{a} \quad \text{(width of diffraction peak from min to min)} \]
Two single slit diffraction patterns are shown. The distance from the slit to the screen is the same in both cases. Which of the following could be true?

(a) The slit width \( a \) is the same for both; \( \lambda_1 > \lambda_2 \).
(b) The slit width \( a \) is the same for both; \( \lambda_1 < \lambda_2 \).
(c) The wavelength is the same for both; width \( a_1 < a_2 \).
(d) The slit width and wavelength is the same for both; \( p_1 < p_2 \).
(e) The slit width and wavelength is the same for both; \( p_1 > p_2 \).
Combined Diffraction and Interference

So far, we have treated diffraction and interference independently. However, in a two-slit system both phenomena should be present together.

\[
I_{2\text{slit}} = 4I_{1\text{slit}} \cos^2 (\delta) \left( \frac{\sin (\alpha)}{\alpha} \right)^2;
\]

\[
\alpha = \frac{\pi a}{\lambda L} \quad y = \frac{\pi a}{\lambda} \sin \theta;
\]

\[
\delta = \frac{\pi d}{\lambda L} \quad y = \frac{\pi d}{\lambda} \sin \theta.
\]

Notice that when \(d/a\) is an integer, diffraction minima will fall on top of “missing” interference maxima.
Circular Apertures

When light passes through a **circular** aperture instead of a vertical slit, the diffraction pattern is modified by the 2D geometry. The minima occur at about $1.22 \frac{\lambda}{D}$ instead of $\frac{\lambda}{a}$. Otherwise the behavior is the same, including the spread of the diffraction pattern with decreasing aperture.
The Rayleigh Resolution Criterion says that the minimum separation to separate two objects is to have the diffraction peak of one at the diffraction minimum of the other, i.e., $\Delta \theta = 1.22 \frac{\lambda}{D}$.

**Example:** The Hubble Space Telescope has a mirror diameter of 4 m, leading to excellent resolution of close-lying objects. For light with wavelength of 500 nm, the angular resolution of the Hubble is $\Delta \theta = 1.53 \times 10^{-7}$ radians.
A spy satellite in a 200km low-Earth orbit is imaging the Earth in the visible wavelength of 500nm.

How big a diameter telescope does it need to read a newspaper over your shoulder from Outer Space?
Δθ = 1.22 λ/D

Letters on a newspaper are about Δx = 10mm apart. Orbit altitude R = 200km & D is telescope diameter.

Christine’s Favorite Formula:

Δx = RΔθ = R(1.22λ/D)

D = R(1.22λ/Δx)

= (200x10^3 m)(1.22x500x10^{-9} m)/(10x10^{-3} m)

= 12.2 m