8.3 Quantum Trajectory

The optical Bloch Eqs. describe behavior of a collection or "ensemble" of atoms. For example, when we say a two-level atom suffers 50% loss we mean that in an ensemble 50% of the atoms are missing an electron from the two levels under consideration. One atom cannot be missing ½ an electron. A measurement made on a single atom shows either the electron is there (50%) or not (50%). Quantum trajectory approach was developed to study loss and decoherence in individual quantum systems.

Quantum Jumps

Consider a three-level atom isolated in empty space. For concreteness let's consider hydrogen:

\[ |1\mathrm{~s}\rangle = |1\rangle \quad |1\mathrm{~s}\rangle = |2\rangle \]

\[ \text{UV} \quad \lambda = 121.5 \text{nm} \]

The transition \( |2\mathrm{p}\rangle = |2\mathrm{~s}\rangle \rightarrow |1\mathrm{~s}\rangle = |1\rangle \) is dipole allowed \( \Delta l = 1 \Delta m = \pm 1 \) and so an atom in \( |2\mathrm{p}\rangle \rightarrow |1\mathrm{~s}\rangle \) by spontaneous emission in \( |1\mathrm{~s}\rangle \).
However \( |2S\rangle = |200\rangle \rightarrow |1S\rangle = |100\rangle \) is not dipole allowed and must take place by a metastable 2-photon transition. Lifetime of \( |2S\rangle \) is 1s instead of 1ns!

I now tune my UV laser at 121.5 nm to the \( |1S\rangle \rightarrow |2P\rangle \) transition

\[
\begin{array}{c}
|2P\rangle \\
\downarrow \text{121.5 nm} \\
|2S\rangle \\
\downarrow \text{121.5} \\
|1S\rangle
\end{array}
\]

The electron will cycle between \( |1S\rangle \rightarrow |2P\rangle \) and also emit 121.6 nm photons by spontaneous emission or stimulated emission — typically \( 1 \text{ photon/}\text{ns} \Rightarrow 10^6 \text{ photons/sec} \). This can be seen with a sensitive CCD camera!

Now we place a second UV beam tuned to 121.5 nm; the \( |1S\rangle \rightarrow |2S\rangle \) will

Every once in a while the electron will jump \( |1S\rangle \rightarrow |2S\rangle \) and stay there for a second. The emission on the \( |1S\rangle \rightarrow |2P\rangle \) will stop.
Here in such a setup we can see the atom blink on and off. Hence we can see quantum jumps.

The unitary evolution of the three-level system in the two laser beams experience "jumps" due to random spontaneous emission events. Recall our theory Ch. 4.3 of atom interacting with a quantized field

\[ P_{g \rightarrow e} \propto n \quad \text{stimulated absorption} \]

\[ P_{e \rightarrow g} \propto n + 1 \quad \text{stimulated emission} \]

The probability of spont. Emission

\[ P_{eg}^{sp} = \left| \mathcal{E}_{ab} \cdot \vec{E}_0 \right|^2 \]

where \( \vec{E}_0 = \frac{i(h \omega)}{2 \epsilon_0 V} e^\frac{-i}{\epsilon_0 V} \) is vacuum electric field.
8.3 Quantum Jumps and Master Eq.

Suppose we have a high-Q cavity with a coherent state $|\psi\rangle$ in the cavity. Periodically, the cavity will emit a photon through the wall at a rate

$$\gamma = \frac{L}{cQ}$$

At the single photon level, this is not a continuous process.

Eventually, $|\psi\rangle \rightarrow |0\rangle$ and only vacuum is left. If $|\psi(0)\rangle$ is a pure state, it will be a mixed state at $t > 0$.

1. $A_P = \gamma \langle \psi | \hat{n} | \psi \rangle \Delta t$
   - $\uparrow$ $\uparrow$ $\uparrow$
   - prob of loss rate of loss # photons in cavity

2. Choose a random number $0 \leq \gamma \leq 1$

3. If $A_P > \gamma$, jump condition
   - $\Rightarrow 1|\psi\rangle \rightarrow 1|\psi_{\text{emitted}}\rangle = \frac{\hat{a}}{\sqrt{\langle \psi | \hat{n} | \psi \rangle}} 1|\psi\rangle$
   - photon annihilation
4. If $\Delta \rho \ll 1$ (no emission) no jump condition

$$1\psi \rightarrow 1\psi_{\text{no}} = \frac{e^{-i\frac{\Delta t}{\hbar} \hat{H}_{\text{eff}}}}{\sqrt{\langle \psi | 1 - \Delta \rho \hat{n} | \psi \rangle}} \langle \psi | \hat{n} | \psi \rangle \left\langle \psi | \hat{H}_{\text{eff}} | \psi \rangle \right\rangle $$

where

$$\hat{H}_{\text{eff}} = \hat{H}_0 - i\frac{\hbar}{2} \hat{a} \hat{a}^\dagger \hat{a} = \hbar \omega \hat{a}^\dagger \hat{a} - i\frac{\hbar}{2} \hat{a}^\dagger \hat{a}$$

$$\hat{H}_{\text{eff}}^\dagger \neq \hat{H}_{\text{eff}} \quad \text{no-hermitian} \quad \omega \rightarrow \omega - i\frac{\chi}{c}$$

gives correct Lorentzian line shape.

$$[\hat{H}_{\text{eff}}, \hat{H}_{\text{eff}}] = 0 \quad \text{and} \quad \hat{H}_{\text{eff}}^\dagger \hat{H}_{\text{eff}} - \hat{H}_{\text{eff}} = i\frac{\hbar}{2} \hat{a}^\dagger \hat{a}$$

$$\Rightarrow \exp[-i\frac{\hbar}{2} \hat{H}_{\text{eff}}^\dagger] \exp[-i\frac{\hbar}{2} \hat{H}_{\text{eff}}] = \exp[-\gamma \Delta t \hat{a}^\dagger \hat{a}]$$

If $\Delta t \ll \gamma = \frac{1}{c \Delta \rho}$

$$\sim \left[ 1 - \gamma \Delta t \hat{a}^\dagger \hat{a} \right]$$

$$\Rightarrow 1\psi_{\text{no}} \sim \frac{e^{-i\frac{\Delta t}{\hbar} \hat{H}_{\text{eff}}}}{\sqrt{1 - \Delta \rho}} \langle \psi | 1 - \gamma \Delta t \hat{a}^\dagger \hat{a} | \psi \rangle$$

$$\sim \left[ 1 - i\gamma \Delta t \hat{a}^\dagger \hat{a} \right] |\psi\rangle \sqrt{\frac{1}{1 - \Delta \rho}}$$

$$\sim \left[ 1 - i\gamma \Delta t \hat{H}_0 - \frac{\gamma \Delta t}{c} \hat{n} \right] |\psi\rangle$$

where

$$\Gamma = \omega - i\chi/2$$

$$\hat{n} = \hat{a}^\dagger \hat{a}$$
The density of \( \hat{\rho}(t) \) becomes

\[
\hat{\rho}(t) = 1 \psi \langle \psi | \rightarrow 1 \psi(\delta t) \langle \psi | \psi(\delta t) \rangle = \rho(\delta t)
\]

\[
= \Delta P \sqrt{\psi_{\text{em}} \rangle \langle \psi_{\text{em}} |} + (1 - \Delta P) \sqrt{\psi_{\text{no}} \rangle \langle \psi_{\text{no}} |}
\]

Jump \hspace{1cm} \text{no jump}

Note

\[
\Delta P \sqrt{\psi_{\text{em}} \rangle \langle \psi_{\text{em}} |} = \left( \frac{\hat{a}^\dagger \psi \langle \psi | \hat{a}^\dagger}{\sqrt{\psi}} \right) \Delta P
\]

\[
= \hat{a}^\dagger \psi \langle \psi | \hat{a}^\dagger \left[ \gamma \langle \gamma | \delta \xi \right]
\]

\[
= \left[ \gamma (\hat{a}^\dagger \psi \langle \psi | \hat{a}^\dagger) \delta t \right] \hat{\rho} = \gamma \delta t \hat{a}^\dagger \hat{a}^\dagger
\]

\[
(1 - \Delta P) \sqrt{\psi_{\text{no}} \rangle \langle \psi_{\text{no}} |} = \left( \frac{\hat{a}^\dagger \psi \langle \psi | \hat{a}^\dagger}{\sqrt{\psi}} \right) (1 - \Delta P)
\]

\[
= \hat{a}^\dagger \psi \langle \psi | \hat{a}^\dagger \left[ i \phi \delta t H_0 - \frac{\gamma}{2} \delta t \hat{\eta} \right]
\]

\[
+ [-i \phi \delta t H_0 - \frac{\gamma}{2} \delta t \hat{\eta}] \hat{\rho} \cdot 1 + \mathcal{O}(\delta t^3)
\]

\[
= \rho(t) - i \phi \delta t \left[ H_0, \hat{\rho} \right] - \frac{\gamma}{2} \delta t \left[ \hat{\eta} \hat{\rho} + \hat{\rho} \hat{\eta} \right]
\]

\[
\Rightarrow \rho(t + \delta t) - \rho(t) = -i \phi \left[ H_0, \hat{\rho} \right] + \frac{\gamma}{2} \left[ \hat{\eta} \hat{\rho} + \hat{\rho} \hat{\eta} \right]
\]

\[
\Rightarrow \frac{i \hbar}{\delta t} \frac{d\hat{\rho}(t)}{dt} = \left[ \hat{H}_0, \hat{\rho} \right] + \frac{\gamma}{2} \left[ \hat{a} \hat{a}^\dagger \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{a} \hat{\rho} \right]
\]

\text{Schrödinger} \hspace{2cm} \text{Louvillian Class}

\text{MASTER EQ.}
The super operator

\[
\hat{L}(\rho) = \frac{\hbar}{2} \left[ 2 \hat{a} \hat{a}^\dagger - \hat{n} \rho - \rho \hat{n} \right]
\]

is called the *Liouville–Loss operator*.

In general the master equation has to be integrated numerically.

However for \( \delta t \ll \hbar \) [no jump] let \( |\psi\rangle \equiv |k\rangle \) and \( \hat{H}_{\text{eff}} = \hbar (\omega - i \gamma / 2) \hat{a}^\dagger \hat{a} \)

\[
|\psi_{\text{no}}\rangle = \frac{e^{-i \delta t \hat{H}_{\text{eff}}}}{\sqrt{\langle k | e^{-i \delta t \hat{H}_{\text{eff}}} | k \rangle}} |k\rangle = \frac{e^{-i (\omega - i \gamma / 2) \delta t}}{\sqrt{\langle k | e^{-i \delta t \hat{H}_{\text{eff}}} | k \rangle}} |k\rangle
\]

using \( e^{\hat{A}} |k\rangle = |\langle k | e^{\hat{A}} \rangle \rangle \)

\[
|\psi_{\text{no}}\rangle = \frac{|\langle k | e^{-i \delta t \hat{H}_{\text{eff}}} \rangle \rangle}{\sqrt{\langle k | e^{-i \delta t \hat{H}_{\text{eff}}} \rangle \langle k |}} \approx |k\rangle \text{ if } \delta t \ll \gamma
\]

\[
\approx |\langle k | e^{-i \omega \delta t - \frac{\gamma}{2} \delta t^2} \rangle \rangle
\]

\[\uparrow \quad \uparrow\]

*normal loss evolution*

so effect is *dissipative exponential decay*.\]
General Solution

Let $|\psi(0)\rangle = |x\rangle = \sum_n C_n(0) |n\rangle$

$$C_n(0) = e^{-\frac{n+1}{2}} \frac{z^n}{\sqrt{n!}}$$

$$p_x = |\langle x|x\rangle| = \sum_{nm} C_n^{\ast}(0) C_m(0) \langle n| \langle n| = e^{-\sum_{nm} \frac{z^n z^m}{\sqrt{n! m!}} C_n^{\ast}(0) C_m(0)}$$

$$p(t) = |\psi(t)\rangle \langle \psi(t)| = \sum_{nm} C_n(t) \langle n| \langle n|$$

$$\dot{p}(t) = \sum_{nm} \dot{C}_n(t) \langle n| \langle n|$$

$$[H_0, p] = \hbar \omega \sum_{nm} C_n(t) \left[ \hat{n}, \langle n| \langle n| \right]$$

$$\quad = \hbar \omega \sum_{nm} C_n(t) \langle n-m| \langle n|$$

$$a \rho a^+ = \sum_{nm} C_n(t) \left[ a \langle n| \langle n-1| \langle n| a^+ \right]$$

$$\quad = \sum_{nm} C_n(t) \langle n-1| \langle n+1|$$

$$\hat{n} \rho = \sum_{nm} C_n(t) \langle n-m| \langle n|$$

$$\dot{\hat{n}} \rho = \sum_{nm} C_n(t) \langle n-m| \langle n|$$

so $\dot{\hat{p}} = [\hat{H}_0, \hat{p}] + \hat{L}(\rho)$

gives doubly infinite coupled set of linear double - $\omega$s.
Recall $e^{a\hat{n}}|\psi\rangle = 1e^{a}\langle\psi|$ for all $a \in \mathbb{C}$.

Hence no jump

\[
|\psi\rangle \rightarrow e^{-\frac{\Delta t}{2}}|\psi\rangle = \frac{|e^{-\frac{\Delta t}{2}}\psi\rangle}{\langle\psi|e^{-\frac{\Delta t}{2}}\psi\rangle} = |\psi\rangle
\]

\[\bar{n} = |\psi|^2\]

so coherent state remains coherent—just shrinks! This is true in general if for $\bar{n} > 1$.

No jump classical.

\[
|\psi\rangle \rightarrow \text{LOSS} \rightarrow |\psi\rangle e^{-\frac{\Gamma}{2}\tau} = |\psi\rangle e^{-\frac{\Gamma}{2c}L}
\]

\[\tau = \frac{L}{c}\]

where $\Gamma$ is loss per unit length.

Beer's Law For Absorption

\[
|\psi\rangle \rightarrow |\psi\rangle e^{-\Gamma L}
\]
Recall

\[ |\text{even}\rangle = N_e |x\rangle \pm |\bar{x}\rangle \]

\[ |\text{odd}\rangle = N_o |x\rangle \mp |\bar{x}\rangle \]

are the even and odd cats.

\[ \hat{A}^2 |\text{cat}\rangle = A^2 |\text{cat}\rangle \] so \text{cat} is

\underline{immune to two jumps}

\[ \hat{A} |\text{even}\rangle = |\text{odd}\rangle \]
\[ \hat{A} |\text{odd}\rangle = |\text{even}\rangle \]

Hence jump causes cats to switch
from even to odd.

\[ N_o \text{ jump} \]

\[ e^{-\frac{\xi}{2} \hat{A}^2 t} |\text{cat}\rangle = N \left[ |x e^{-\frac{\xi}{2} \hat{A}^2 t} \rangle \pm |\bar{x} e^{-\frac{\xi}{2} \hat{A}^2 t} \rangle \right] \]

causes cat to shrink.

\[ \text{Compare Wigner Fns Eq. 7.137/17.138} \]

\[ W_e (x,y) = \frac{1}{\pi \left[ 1 - e^{2\xi^2} \right]} \left\{ -2 [e^{2\xi^2}] e \uparrow \right. \]
\[ \left. -2 \exp \left[ -2 (x^2 + y^2) \right] \cos 4y \alpha \right\} \]

\[ W_0 (x,y) = \frac{1}{\pi \left[ 1 - e^{2\xi^2} \right]} \left\{ -2 \exp \left[ -2 (x^2 + y^2) \right] \sin 4y \alpha \right\} \]
Since $x, \bar{x} \in \mathbb{R}$ we can plot slice at $x_g = 0$

We $[x=0,y]$ 
\[ w_0 \] 
\[ w_0 \] 

On each jump

the wigglers tend to cancel out

\[ \lim_{t \to \infty} \]

No wigglers $\Rightarrow$ classical mixture

\[ \hat{\rho}_{\text{mix}} = |x e^{-\frac{2}{\hbar}t} \rangle \langle e^{\frac{2}{\hbar}t} | + | \bar{x} e^{-\frac{2}{\hbar}t} \rangle \langle e^{\frac{2}{\hbar}t} | \]

No more interference. As $t \to \infty$

\[ \hat{\rho} \to 10 \left| 0 \right> \left< 0 \right| \]

Pure vacuum state!
For full three-level system we must solve 3x3 density matrix equations. Assume \( |\Psi(0)\rangle \) a pure state:

\[
|\Psi\rangle = C_{10}|10\rangle + C_{11}|11\rangle + C_{12}|12\rangle
\]

\[
\hat{\rho}_{pure} = |\Psi\rangle \langle \Psi| = \begin{bmatrix}
C_{10}^* C_{10} & C_{10}^* C_{11} & C_{10}^* C_{12} \\
C_{11}^* C_{10} & C_{11}^* C_{11} & C_{11}^* C_{12} \\
C_{12}^* C_{10} & C_{12}^* C_{11} & C_{12}^* C_{12}
\end{bmatrix} \\
= \begin{bmatrix}
P_{10} & P_{10} & P_{20} \\
P_{11} & P_{11} & P_{21} \\
P_{12} & P_{12} & P_{22}
\end{bmatrix}
\]

where

\[
P_{ii} = |C_i|^2 = P_i \quad \text{and} \quad \sum_{i=0}^{2} P_{ii} = 1
\]

cons of probability

\( P_{ii} \) is called a population in level \( i \rangle \)

\( \gamma_{ij} \) (off diagonal) \( \gamma_{ij} = \gamma_{ji} \) coherence terms

Let us assume \( A_{10} = A_{20} = 0 \)

How do we obtain 3-level Rabi solution?

TDSE:

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle
\]

\[
-i\hbar \frac{d}{dt} \langle \psi | = \langle \psi | \hat{H}
\]
\[ i \frac{d\psi}{dt} = i \left[ \psi \times \psi^\dagger + \psi^\dagger \times \psi \right] \]
\[ = \hat{\mathbf{A}} \psi \times \psi^\dagger - \psi^\dagger \psi \times \psi \]
\[ = [\hat{\mathbf{A}}, \rho] \]

Even without loss / dissipation, \( A_{10} = A_{20} = 0 \)
this is NINE coupled first order diffy-Qs, even assuming field is classical. Adding loss
\[ i \frac{d\hat{\rho}}{dt} = [\hat{\mathbf{A}}, \hat{\rho}] + \hat{\mathbf{W}} [\hat{\rho}] \]

Just makes it worse - must be done numerically. However, there is another approach!

Assume \( A_{10} = A_1 \neq 0 \) and \( A_{20} = A_2 \neq 0 \)
\( B_{10} = B_{01} = B_1 \) Reciprocity
\( B_{20} = B_{02} = B_2 \) Reciprocity

For long times / after many jumps / off diagonal coherence terms will vanish
\( t \to 0 \Rightarrow \rho_{ij} = 0 \quad i \neq j \)
Hence we can write down rate equations for only \( \rho_{11} = \rho_1 \)
\( \rho_{22} = \rho_2 \)
\( \rho_{33} = \rho_3 \)
We assume system is closed, so population is conserved: \( p_{ii}(t) + p_{22}(t) + p_{33}(t) = 1 \) for all \( t \). That is, electron is always in one of three levels. We now just a la Einstein compute the rates.

\[
\dot{p}_1 = -[A_1 + B_1I_1]p_1 + B_1I_1p_0
\]

\[
\text{spontaneous emission prob } A_1p_1 \text{ is proportional to Einstein A and prob. } p_1 = p, \text{ that electron is in level } 11\rangle. \text{ Similarly stimulated } < BI_1
\]

\[
\dot{p}_2 = -[A_2 + B_2I_2]p_2 + B_2I_2p_0
\]

Finally, cons. prob.

\[
\dot{p}_{00} = -[B_1I_1 + B_2I_2]p_{00} + [A_1 + B_1I_1]p_{11} + [A_2 + B_2I_2]p_{22}
\]

\text{st. abs.}\]

Let \( A_i = \lambda_i \) and fundamental \( B_iI_i = \beta_i \) rate equations are
\[ \begin{align*}
\dot{\rho}_0 &= -[\beta_1 + \beta_2] \rho_0 + [\chi_1 + \beta_1] \rho_1 + [\chi_2 + \beta_2] \rho_2 \\
\dot{\rho}_1 &= -[\chi_1 + \beta_1] \rho_1 + \beta_1 \rho_0 \\
\dot{\rho}_2 &= -[\chi_2 + \beta_2] \rho_2 + \beta_2 \rho_0 \\
1 &= \rho_1 + \rho_2 + \rho_3
\end{align*} \]

First consider steady state solutions
\[
\dot{\rho}_2 = 0 \quad \text{for} \quad t \to \infty
\]

\[ \begin{align*}
0 &= -[\beta_1 + \beta_2] \rho_0 + [\chi_1 + \beta_1] \rho_1 + [\chi_2 + \beta_2] \rho_2 \\
0 &= -[\chi_1 + \beta_1] \rho_1 + \beta_1 \rho_0 \\
0 &= -[\chi_2 + \beta_2] \rho_2 + \beta_2 \rho_0 \\
1 &= \rho_0 + \rho_1 + \rho_2
\end{align*} \]

These can be solved exactly!
But first consider some limiting cases

\[ \beta_2 = \chi_2 \approx 0 \quad \text{That is level two is decoupled} \]
\[ \chi_2 \approx \text{very weak or off} \]
Also \( \chi_1 \approx 0 \)
\( \chi_1 \ll \beta_1 \) so spontaneous.

Neg.
\[ \begin{align*}
0 &= -\beta_1 \rho_0 + \beta_1 \rho_1 \quad \Rightarrow \quad \rho_1 &= \rho_0 \\
0 &= -\beta_2 \rho_2 + \rho_1 \\
1 &= \rho_0 + \rho_1 + \rho_2 \\
\Rightarrow \quad \rho_0 &= \rho_1 = \frac{1}{2} \quad \text{if population initially} \\
\rho_2 &= 0 \\
\rho_0 &= \frac{1}{2} \quad \text{Detailed Balance}
\end{align*} \]
Assume: \( \chi_1 \equiv \chi_2 \approx 0 \)

or more realistically \( \chi_1 \ll \beta_1 \) and \( \chi_2 \ll \beta_2 \)

In this case \( \rho_0(0) = 1 \); \( \rho_1(0) = 0 \); \( \rho_2(0) = 0 \)

Initial conditions.

\[
\begin{align*}
0 &= -[\beta_1 + \beta_2] \rho_0 + \beta_1 \rho_1 + \beta_2 \rho_2 \\
0 &= -\beta_1 \rho_1 + \beta_1 \rho_0 \quad \Rightarrow \quad \rho_0 = \rho_1 \\
0 &= -\beta_2 \rho_2 + \beta_2 \rho_0 \quad \Rightarrow \quad \rho_0 = \rho_2 = \rho_1 \\
1 &= \rho_0 + \rho_1 + \rho_2 \\
\Rightarrow \; t \to \infty \quad \boxed{\rho_0 = \rho_1 = \rho_2 = \frac{1}{3}}
\end{align*}
\]

Again, detailed balance places population equally. Spontaneous Emission \( \chi_1 \), \( \chi_2 \) spoils detailed balance! Hence \( \chi_1 > \beta_1 \) and \( \chi_2 < \beta_2 \) we expect

\[
\begin{align*}
\rho_0(0) &< \frac{1}{3} \\
\rho_1(0) &< \frac{1}{3} \\
\rho_2(0) &< \frac{1}{3}
\end{align*}
\]

as electron comes down faster than it goes up.
Astonishingly if you plug $t \to \infty$ equations into mathematical it produces an exact solution

\[
\begin{align*}
\rho_1(t) &= \frac{(x_1 + \beta_1) \beta_2}{x_1 (x_2 + 2 \beta_2) + \beta_1 (2 x_2 + 3 \beta_2)} \\
\rho_2(t) &= \frac{(x_1 + \beta_1) \beta_2}{x_2 (x_1 + 2 \beta_1) + \beta_2 (2 x_1 + 3 \beta_1)} \\
\rho_{00}(t) &= 1 - \rho_1(t) - \rho_2(t)
\end{align*}
\]

which agrees with book. This makes no assumptions about relative sizes of $\beta_1 \beta_2$ as indicated that $\rho_1 \leftrightarrow \rho_2$ under interchange of $1 \leftrightarrow 2$.

Taking $x_2 = \beta_2 \neq 0$ gives $\rho_0 = \rho_1 = \frac{1}{2}$; $\rho_2 = 0$ let's do this carefully

(\text{I}) \quad x_1 = x_2 = 0

\[
\begin{align*}
\rho_1(t) &= \frac{\beta_1 \beta_2}{2 \beta_1} = \frac{1}{2} \\
\rho_2(t) &= \frac{\beta_1 \beta_2}{3 \beta_1 \beta_2} = \frac{1}{3}
\end{align*}
\]

\[
\rho_{00}(t) = \frac{1}{2} \quad \text{Independent of } \beta_1 \gg \beta_2 \text{ or } \beta_1 \ll \beta_2
\]

(\text{II}) \quad \beta_2 = 0 \quad x_1 = 0

\[
\begin{align*}
\rho_1(t) &= \frac{x_2 \beta_1}{2 x_2 \beta_1} = \frac{1}{2} \\
\rho_2(t) &= 0
\end{align*}
\]

\[
\rho_{00}(t) = \frac{1}{2}
\]
Quantum Jump Conditions

\[ \text{Strong I}_1 \text{ pump } \Rightarrow \beta_1 \gg \lambda_1 \text{ we can take } \lambda_1 \text{ negligible} \]

\[
\rho_1(\infty) = \frac{(\lambda_2 + \beta_2)\beta_1}{(2\lambda_2 + 3\beta_2)\beta_1} = \frac{\lambda_2 + \beta_2}{2\lambda_2 + 3\beta_2} \]

\[ \text{IND OF } \beta_1 \]

\[
\rho_2(\infty) = \frac{\beta_1 \beta_2}{2\lambda_2 \beta_1 + 3\beta_2 \beta_2} = \frac{\beta_2}{2\lambda_2 + 3\beta_2} \]

which agrees with book. Note this holds \( \forall \beta_2 \) and so even if \( \beta_2 \ll \beta_1 \) (weak \( \text{I}_2 \text{ pump} \)) there is probability to find electron in \( \text{l}_2 \rangle \)

We can further assume \( \beta_2 \gg \lambda_2 \Rightarrow \gamma_2 = \frac{\lambda_2}{\beta_2} < 4 \)

\[
\rho_1(\infty) = \frac{\gamma_2 + 1}{2\gamma_2 + 3} \approx \frac{1}{3} + \frac{1}{9} \gamma_2 \gg \frac{1}{3} \]

\[
\rho_2(\infty) = \frac{1}{2\gamma_2 + 3} \approx \frac{1}{3} - \frac{2}{9} \gamma_2 \ll \frac{1}{3} \]

So even when \( \text{I}_1 \text{ pump is very strong} \) the prob of finding in \( \text{l}_2 \rangle \text{ is very high} \)
For \( \alpha \ll 20 \) we must solve

\[
\begin{align*}
\dot{\rho}_1 &= - [\alpha + \beta_1] \rho_1 + \beta_1 (1 - \rho_1 - \rho_2) \\
\dot{\rho}_2 &= - [\alpha + \beta_2] \rho_2 + \beta_2 (1 - \rho_1 - \rho_2)
\end{align*}
\]

where we have eliminated \( \rho_0 = 1 - \rho_1 - \rho_2 \)

\[
\begin{align*}
\dot{\rho}_1 &= - [\alpha + 2 \beta_1] \rho_1 + \beta_1 - \beta_1 \rho_2 \\
\dot{\rho}_2 &= - [\alpha + 2 \beta_2] \rho_2 + \beta_2 - \beta_2 \rho_1
\end{align*}
\]

with initial condition \( \rho_1(0) = \rho_2(0) = 0 \) \( \rho_0(0) = 1 \)

Mathematics gives: A very large output!

However solution is exact! We will have to make some approximations. Let's take \( \beta_1 \gg \alpha, \alpha_0 \) as before we may write

\[
\begin{align*}
\frac{1}{\rho_1 \rho_2} \dot{\rho}_1 &= - \left[ \frac{\alpha}{\rho_1 \rho_2} + \frac{\beta_1}{\rho_2} \right] \rho_1 + \frac{1}{\beta_2} - \frac{1}{\rho_2} \rho_2 \\
\frac{1}{\rho_1 \rho_2} \dot{\rho}_2 &= - \left[ \frac{\alpha}{\rho_1 \rho_2} + \frac{\beta_2}{\rho_2} \right] \rho_2 + \frac{1}{\beta_1} - \frac{1}{\beta_2} \rho_1
\end{align*}
\]

\[
\begin{align*}
\dot{\rho}_1 &= - 2 \beta_1 \rho_1 + \beta_1 - \beta_1 \rho_2 \\
\dot{\rho}_2 &= - 2 \beta_2 \rho_2 + \beta_2 - \beta_2 \rho_1
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\beta_2} \dot{\rho}_1 &= - 2 \frac{\beta_1}{\beta_2} \rho_1 + \frac{\beta_1}{\beta_2} - \frac{\beta_1}{\beta_2} \rho_2 \\
\frac{1}{\beta_2} \dot{\rho}_2 &= - 2 \rho_2 + 1 - \rho_1
\end{align*}
\]

Inserting \( \rho_2 / \beta_2 \rightarrow \frac{1}{\beta_2} \rho_1 \) and drop \( \frac{\beta_1}{\beta_2} \ll 1 \)
\[ \frac{1}{\beta_2} \dot{\rho}_1 = -2 \frac{\beta_1}{\beta_2} \rho_1 + \frac{\beta_1}{\beta_2} - \frac{\beta_1}{\beta_2} \left[ -2 \rho_1^2 + 1 - \rho_1 \right] \]

This tells us

\[ \dot{\rho}_1 = -\beta_1 \rho_2 \quad \rho_1(0) = 0 \]

\[ \rho_2 = -2 \beta_2 \rho_2 + \frac{\beta_2}{\beta_1} - \frac{\beta_2}{\beta_1} \rho_1 \quad \rho_2(0) = 0 \]

After some work in *Mathematica* I get

\[ \rho_1(t) \approx \frac{1}{3} \left[ 1 + \frac{1}{t} \left( e^{-3 \beta_2 t/2} - 3 e^{-2 \beta_1 t} \right) \right] \]

\[ \rho_2(t) \approx \frac{1}{3} \left[ 1 - e^{-3 \beta_2 t} \right] \]

\[ \rho_1(0) = \frac{1}{3} \left[ 1 + \frac{1}{0} (1-3) \right] = \frac{1}{3} \left[ 1-1 \right] = 0 \quad \checkmark \]

\[ \rho_2(0) = \frac{1}{3} \left[ 1 - 1 \right] = 0 \quad \checkmark \]

\[ \rho_1 \text{ is } \rho_2 \text{ at } 0 \quad \checkmark \]

so for \( 0 < t \ll T_2 = \frac{1}{\beta_2} \) [Jump Time]

\[ \rho_1(t \ll T_2) \approx \frac{1}{3} \left[ 1 + \frac{1}{t} \left[ 1 - 3 e^{-2 \beta_1 t} \right] \right] \]

\[ = \frac{1}{3} \left[ \frac{3}{2} - \frac{3}{2} e^{-2 \beta_1 t} \right] \rightarrow \frac{1}{2} \quad 0 < 1 \quad \text{Saturation} \]

\[ \rho_1(t \gg T_2) \approx \frac{1}{3} \quad \text{Detailed Balance.} \]
Let $\beta_1 = 1$  $\beta_2 = 0.1$

$\rho_{11}$