It is typical to approximate decoherence in semiclassical theory phenomenologically.

Recall Rabi Model:

\[ E = \hbar \gamma g \quad \omega_0 = \omega_e - \omega_g \quad \text{\(\hbar\omega_0\)} \]

\[ E_e = \hbar \omega_e \]

\[ |1\rangle = C_g(t) e^{-i\omega_g t} |g\rangle + C_e(t) e^{-i\omega_e t} |e\rangle \quad \text{Eq. 4.70} \]

\[ C_g(t) = \frac{iU}{\pi} e^{i\Delta t/2} \sin \left[ \frac{\Omega t}{2} \right] \]

\[ C_e(t) = e^{-i\Delta t/2} \left[ \cos \frac{\Omega t}{2} - i \frac{\Delta}{\Omega} \sin \frac{\Omega t}{2} \right] \]

Let's define \( \Delta = \omega - \omega_0 = 0 \)

\[ Q = \left( \Delta^2 + \Omega^2 \right)^{1/2} \Rightarrow U = -\frac{\hbar}{4} \langle e | \frac{\Delta}{\Omega} | g \rangle = -Q_0 \]

\[ \Rightarrow |1\rangle = e^{-i\omega_g t} |g\rangle + e^{-i\omega_e t} \left[ \cos \frac{\Omega t}{2} |e\rangle + i \sin \frac{\Omega t}{2} |e\rangle \right] \]

We may construct density matrix:

\[ \hat{\rho}(t) = |1\rangle \langle 1| = \begin{bmatrix} 1 & e^{-i\omega_g t} \langle g | e \rangle \\ e^{i\omega_g t} \langle e | g \rangle & 1 \end{bmatrix} \begin{bmatrix} 1 \\ e^{-i\omega_e t} \langle e | g \rangle \end{bmatrix} \]

Recall \( W(t) = \langle ee \rangle - \langle gg \rangle = \sin^2 \Omega t - \cos^2 \Omega t = -2 \sin \Omega t \cos \Omega t \)

is the inversion. The off diagonal terms \( \frac{1}{2} \sin \Omega t \cos \Omega t \) are the coherence terms. These are interference terms. Note:

\[ \hat{\rho}_{\text{mix}} = \begin{bmatrix} \sin^2 \Omega t \cos \Omega t / 2 & 0 \\ 0 & \cos^2 \Omega t \cos \Omega t / 2 \end{bmatrix} \]
Describes a classical statistical mixture,
\[ \rho_e = \frac{1}{2} \rho_{\text{loc}}^2 \quad \rho_g = \frac{1}{2} \rho_{\text{loc}}^2 \]
and so measurement produce these randomly like coin flipping. The off-diagonal terms describe quantum interference.

If the atom is coupled to a heat bath on any system with an \( \infty \) number of modes the coherence will decay with a rate \( \gamma = 1/T \). We model this as
\[ e^{\pm i \omega t} \rightarrow e^{\pm i \omega_0 - \gamma t} \]
and so the off diagonal terms experience exponential decay
\[
\hat{\rho} \rightarrow \hat{\rho}_{\text{mix}}
\]

Note in decoherent probability is still conserved, a measurement will produce \( \pm \) or \( \pm \).
Typical we write \( \nu_i \hat{\rho} = \text{Re} \left[ \rho_y \right] + \text{Re} \left[ \rho_g e^{i \omega t} \right] \)
\[ \Rightarrow \mu = \frac{1}{2} \text{Re} \left[ \rho_y \right] \rho_{\text{loc}} e^{-\omega t} \]
\[ \nu = \frac{1}{2} \text{Re} \left[ \rho_g \right] \rho_{\text{loc}} e^{-\omega t} \]
As \( \mu \) and \( \nu \) decay to zero
\[ \rightarrow \text{towards North/South pole possible.} \]
In this same model we can introduce

\[ \text{loss} \quad \begin{cases} \gamma_e = 1/T_e, & \gamma_g = 1/T_g \end{cases} \quad \text{two same} \]

third level uncoupled from first.

In this case probability is not conserved:

\[ \gamma_t = \gamma_e + \gamma_g \]

\[ P_e = e^{-\gamma_e t} \quad P_g = e^{-\gamma_g t} \]

\[ Pe + Pg \leq 1 \]

Here sphere shrinks.
Let us go back and review:

\[ \omega = \omega_0 \]

\[ v = v_0 \cos \omega_0 t \]

\[ i \frac{\dot{\rho}}{\dot{t}} = [\hat{H}, \hat{\rho}] \]

\[ \Rightarrow i \hbar \dot{\rho} = \begin{bmatrix} \hbar \omega_0 & v \\ -v & \hbar \omega_0 \end{bmatrix} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \]

\[ = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} \hbar \omega_0 & v \\ -v & \hbar \omega_0 \end{bmatrix} \]

\[ \Rightarrow i \hbar \begin{bmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{bmatrix} = \begin{bmatrix} \hbar \omega_0 \rho_{11} + v \rho_{21} & \hbar \omega_0 \rho_{12} + v \rho_{22} \\ v \rho_{11} + \hbar \omega_0 \rho_{21} & v \rho_{12} + \hbar \omega_0 \rho_{22} \end{bmatrix} \]

\[ \frac{d\rho_{22}}{dt} = -\frac{d\rho_{11}}{dt} = -\frac{i}{\hbar} \omega_0 \rho_{12} + \frac{i}{\hbar} \omega_0 \rho_{21} \]

\[ \frac{d\rho_{12}}{dt} = -\frac{d\rho_{21}}{dt} = \frac{i}{\hbar} \omega_0 [\rho_{11} - \rho_{22}] \]

These are also called optical Bloch Equations.

with \( \rho_{11} - \rho_{22} = \omega \) and \( v \cos \omega t = 2 \rho_{12} \)

we recover some results.

However, let's take a simple case of loss.
If \( \omega_0 \) is a frequency standard, we can measure time very accurately by counting peaks.

\[
T = N \frac{2\pi}{\omega_0}
\]

\[
P_e = \frac{\omega_0}{2}\left[ Pe + P_f \right]
\]

\[
T = \frac{2\pi}{\omega_0} - \frac{\omega_0}{2}\left[ Pe + P_f \right]
\]

Atomic clock

Ramsey Interferometer
we allow for spontaneous emission out of upper level into some third level, or coupling to environment phenomenologically we insert decay terms.

\[
\dot{\rho}_{21} = - \dot{\rho}_{11} = - \frac{i}{2} \omega_0 (\rho_{12} - \rho_{21}) - 2 \gamma \rho_{22} \\
\dot{\rho}_{12} = - \dot{\rho}_{21} = \frac{i}{2} \omega_0 (\rho_{12} - \rho_{21}) - \frac{\hbar}{2} \rho_{12}
\]

If atom in empty space \( \gamma = \Gamma = SpE \) emission rate is only source of decohherence. However interaction with environment

\[
\Gamma = \gamma_{pe} + \gamma_{dec} \\
\gamma_{pe} \uparrow \text{ collisions} \uparrow \text{ heat bath} \text{ etc.} \text{ decohherence}
\]

\[
\frac{1}{\gamma} = T_1 \quad \frac{1}{\Gamma} = T_2 \text{ decohherence time}
\]
Exact solution of 2x2 optical Bloch equations with loss/decoherence not possible. Integrate 2x2 coupled 1st order linear diffy-Qs.

Limiting case $\Omega \gg \Gamma$: loss is small

$$\hat{\rho}(t) = \begin{bmatrix}
\sin(\Omega t/2) e^{-\gamma t} & -\frac{i}{\gamma} \sin[\Omega t] e^{-i\omega t} \\
\frac{i}{\gamma} \sin[\Omega t] e^{-i\omega t} & \cos^2(\Omega t/2) e^{-\gamma t}
\end{bmatrix} = \begin{bmatrix}
\rho_{ee} & \rho_{eg} \\
\rho_{ge} & \rho_{gg}
\end{bmatrix}$$

Optical Bloch

$$\mathbf{W} = -\cos[\Omega t] e^{-\gamma t} \quad \text{population loss}$$

$$\mathbf{M} = \sin[\Omega t] e^{-i\omega t} \quad \text{decoherence}$$

$$\mathbf{V} = -\sin[\Omega t] e^{-\gamma t} \quad \text{cannot}$$

$$\text{population loss} \Rightarrow \text{decoherence}$$

$$\text{decoherence} \not\Rightarrow \text{population loss}$$
Special Case $\gamma = 0$ (neg. spontaneous emission)

$\mathbf{\Pi} = \mathbf{0} + \gamma_{\text{DEC}}$ (atomic collisions)

$\mathbf{\Omega} = -\mathbf{c} \mathbf{c}^{\dagger} \mathbf{M} \mathbf{M}^\dagger \mathbf{E}^{-\gamma_{\text{DEC}}} - \gamma_{\text{DEC}}$

$\mathbf{\Pi} = \mathbf{M} \mathbf{M}^\dagger \mathbf{E}^{-\gamma_{\text{DEC}}}$

Decohere

$\rho(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\rho(t) = \begin{bmatrix} 1 - \frac{t}{2} i \omega t e^{-nt i \omega t} \\ -\frac{t}{2} i \omega t e^{-nt i \omega t} + 1 \end{bmatrix}$

$t \to \infty$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Mixed state!

50% up 50% down

$e^{-\gamma_{\text{DEC}} t}$

Ramsey
special case $\zeta_{spe} \to \zeta_{dech} = 0$
only decoherence due to loss!

$P(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Psi$

$P(t) = \begin{bmatrix} e^{-\gamma t} & -\frac{i}{\hbar} e^{i\omega t} e^{-\gamma t} \\ -\frac{i}{\hbar} e^{i\omega t} e^{-\gamma t} & e^{-\gamma t} \end{bmatrix}$

$\downarrow \ t \to \infty$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Eventually population is all lost to inaccessible levels.
SPECIAL CASE

\[ \hat{p} = 0 \]

steady state

can now solve 4 linear algebra eqs.

\[ \rho_{ee} = \rho_{gg} = \frac{1}{4} \omega^2 \frac{\omega_0^2}{\Omega^2 + \frac{1}{2} \omega_0^2 + \Delta^2} \]

\[ \rho_{eg} = \rho_{ge} = \frac{1}{2} i \Omega \omega_0 e^{-i \Delta t} \]

\[ \Delta = \omega_0 - \omega \]

This is Lorentzian as function of \( \Delta \)

\[ \rho_{ee} \]

spectrum gives width in resonance

width of curve

\[ J = \sqrt{\Gamma^2 + \frac{1}{2} \omega_0^2} \]

\[ \Gamma = \text{full width at half maximum} \]

\[ \frac{\text{Natural line width}}{\omega_0 < \omega_0 < \text{power broad}} \]
Recall semiclassical Rabi Model

Field classical \( \frac{|\psi(0)|^2}{\hbar} \)

Atom quantum

\[ \omega_0 = (E_e - E_g) \gamma \]
\[ \Delta = \omega_0 - \omega = \text{detuning} \]

\[ \hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_Z \]

We can derive Rabi solutions in density matrix form

\[ \hat{H}_A = \hbar \omega \left[ 1 e \langle \phi | e | \phi \rangle | g \rangle \right] = \begin{bmatrix} \hbar \omega_0 & \langle e | 1 \rangle \\ \langle 1 | \omega_0 \rangle & 0 \end{bmatrix} | e \rangle \langle e | \\
V_{eg}^* & 0 \end{bmatrix} e \langle \phi | \psi \rangle \]

where \( V_{eg} = V_{ge} = \langle e | \hat{V}_0 | g \rangle = \langle e | -\hat{A} \cdot \vec{E} | g \rangle \)

\( = -\langle e | \hat{V}_0 | g \rangle \\vec{E} = -\vec{A}_{eg} \cdot \vec{E} \) are off diagonal

Hence

\[ \hat{H} = \begin{bmatrix} \hbar \omega_0 & V_{eg} e^{i\omega t} \\ V_{eg} e^{-i\omega t} & \hbar \omega_0 \end{bmatrix} \]

\[ \text{Take } \quad V_{eg} = V_{ge} = \Re \quad \text{Real} \]

Let \( |\psi(t)\rangle = C_g(t) e^{-i\omega t} |g\rangle + C_e(t) e^{-i\omega t} |e\rangle \)

\[ \Rightarrow \quad \hat{\rho}_\psi = |\psi(0)\rangle \langle \psi(0)| \] density operator pure state

\[ = \begin{bmatrix} C_g^2 & C_e^* C_g e^{i\omega t} \\ C_e C_g e^{-i\omega t} & 1 \end{bmatrix} \begin{bmatrix} |C_e|^2 & C_e C_g e^{i\omega t} \\ C_e^* C_g e^{-i\omega t} & 1 \end{bmatrix} = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{bmatrix} \]
Instead of solving \( \frac{\partial \hat{\psi}}{\partial \epsilon} = i \hat{H} \hat{\psi} \) state eq.,
we now solve
\[
i \frac{\partial \hat{\rho}}{\partial \epsilon} = [\hat{H}, \hat{\rho}] \quad \text{density op. eq.}
\]
For pure states these approaches are equivalent!
However, it is much easier to model loss and
dehaerence in density operator approach!

The three Bloch parameters are
\[
W = \rho_{ee} - \rho_{gg} \quad \text{(Inversion)}
\]
\[
2 \rho_{eg} = M + i N \quad \text{Coherece}
\]

We can immediately lift previous solution Eqs. 4.80,
which we take with \( \Delta = 0 \) for
simplicity Eqs. 4.80

\[
C_e = \frac{i \Delta}{2} (e^{i \omega t / 2})
\]
\[
C_g = \frac{i \Delta}{2} (e^{-i \omega t / 2})
\]

We may construct density matrix
\[
\begin{bmatrix}
\rho_{ee} = |C_e|^2 = \frac{1}{2} \left( e^{-i \omega t / 2} \right) e^{-i \omega t} & \text{coherence} & \rho_{eg} = -\frac{i}{2} \Delta \left[ e^{i \omega t / 2} \right] e^{i \omega t} \\
\rho_{ge} = \frac{i}{2} \Delta \left[ e^{-i \omega t / 2} \right] e^{-i \omega t} & \text{population} & \rho_{gg} = |C_g|^2 = \frac{1}{2} \left( e^{i \omega t / 2} \right) e^{i \omega t}
\end{bmatrix}
\]
\[
W = \rho_{ee} - \rho_{gg} = -\frac{\Delta}{2} \left( e^{-i \omega t / 2} \right) \quad \text{and} \quad M + i N = \rho_{eg} \Rightarrow
\]
\[ W = -\cot \frac{\omega t}{\tau} \]

Inversion

\[ M + iN = -\frac{1}{2} \sin \left[ \omega t \right] e^{i\omega t} = -\frac{1}{2} \sin \omega t [i \cos \omega t - \sin \omega t] \]

\[ \Rightarrow M(t) = \frac{1}{2} \sin \omega t \sin \omega t \]

\[ = \frac{1}{2} \sin^2 \omega t \]

and \[ N(t) = -\frac{1}{2} \sin \omega t \cos \omega t \]

\[ = -\frac{1}{2} \sin \omega t \cos \omega t \]

Recall this is solution assumes \( |\psi(0)\rangle = 1g \)

We construct a Bloch sphere

of radius 1

The vector

\[ [U(t), V(t), W(t)] \]

has length

\[ u^2 + v^2 + w^2 \]

\[ = \cos^2 \omega t + \sin^2 \omega t + \sin^2 \omega t \cos^2 \omega t \]

\[ = \cos^2 \omega t + \sin^2 \omega t [\cos^2 \omega t + \sin^2 \omega t] \]

\[ = \cos^2 \omega t + \sin^2 \omega t \]

\[ = 1 \]

So vector traces out evolution of state on sphere. \( \omega t \) is in equator, \( -\omega t \) is polar.

\[ \omega t = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} \text{ pulse} \]

\[ [u, v, w] = \left[ \sin \omega t, -\cos \omega t, 0 \right] \]
To obtain the probability/time or rate we

take Eq. 4.57

\[ P_{\text{em}}(t) = \left| \frac{\hbar}{\pi} \langle e| \hat{H}_{\text{int}} | g \rangle \cdot E_0 \cdot \frac{\sqrt{\omega_0 + \omega} t}{\sqrt{\omega_0 + \omega_0}} \right|^2 \]

where we take \( n = 0 \) so SP. EM. only

This formula was derived in free space

but for one mode \( \omega_0 \),

we can immediately use Fermi's golden rule

\[ W_{\text{em}} = \frac{2 \pi \hbar}{\hbar^2} \langle e| \hat{H}_{\text{int}} | g \rangle \cdot 1^2 \rho(\omega) \]

where \( \hat{H}_{\text{int}} = -\hat{\mathbf{A}} \cdot \mathbf{E}_0 \) and \( \sqrt{\rho(\omega)} \) is the free space density of modes evaluated at \( \omega = \omega_0 \) from Eq. 2.75 \( \omega \to \omega_0 \)

\[ \rho(\omega_0) = \frac{\omega_0^2}{\pi^2 c^3} \]

\[ S_{\text{em}} = \frac{\pi \hbar^2}{2 \cdot e_0 \cdot n \cdot \omega_0} \]

\[ W_{\text{em}} = \frac{2 \pi \hbar}{\hbar^2} \frac{\omega_0^2}{\pi^2 c^3} \left| \text{deg} \right|^2 \]

\[ = \frac{\omega_0^3}{\pi^2 c^3 \hbar} \left| \text{deg} \right|^2 \]

\[ = A = \frac{\omega_0^3}{2} \text{ Einstein A} \]

Wigner - Weisskopf

For Hydrogen \( |2p\rangle \rightarrow |1s\rangle \)

\[ \text{deg} = e^2 |\langle 2p | r | 1s \rangle|^2 = \frac{e^2}{a_0^2} \]

\[ \text{Bar} - \text{radius} \]

Notice that \( A = \frac{\omega_0^3}{2} \)

UV transitions decay faster than IR.
so now we know the rate \( 125\rightarrow 12S \)

\[ T = \frac{1}{A} \approx 10 \text{ ns} \quad \text{ (fast).} \quad 12S \rightarrow 12S \]

is much harder to compute (second order perturb) but it is much slower (1s).

In this treatment atom is quantum system and vacuum modes is reservoir \((T=0)\). Could do \(T>0\) (heat bath).

Idea is that loss/decoherence occurs when a small quantum system with a few modes is coupled to a large quantum system with many modes.

\[ \text{Analogy coupled pendula} \]

Energy sloshes back and forth

\[ \text{If infinite number energy never returns to } A \text{ again!} \]
Quantum jumps in the photon field

\[ H_R = \sum_k \hbar \omega_k \hat{b}_k \hat{b}_k^\dagger \]

Let us suppose some photon state \( |\Psi\rangle \) is in the cavity/single mode. If \( Q \neq 0 \) every once and a while a photon is lost. The vacuum modes outside the cavity are the reservoir.

\[ \hat{H} = \hat{H}_{\text{cav}} + \hat{H}_{\text{res}} + \hat{H}_{\text{int}} \]

\( \hat{H}_{\text{cav}} = \hbar \omega_0 \) a single mode

\[ \hat{H}_{\text{res}} = \sum_k \hbar \omega_k \hat{b}_k^\dagger \hat{b}_k \text{ all modes} \]

\[ \hat{H}_{\text{int}} = \hbar \sum_k g_k \left[ \hat{a}^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{a} \right] \]

\( g_k = \) coupling constant of cavity to outside

Let's assume \( g_k = g \) \( \forall k \)

We can work in Heisenberg Picture

\[ \dot{\hat{a}} = \frac{i}{\hbar} \left[ \hat{H}, \hat{a} \right] = -i \omega_0 \hat{a}(t) - ig \sum_k \hat{b}_k(t) \]

\[ \dot{\hat{b}}_k = \frac{i}{\hbar} \left[ \hat{H}, \hat{b}_k \right] = -i \omega_k \hat{b}_k(t) - ig \hat{a}(t) \]

Note \( [a, b^\dagger] = 0 \) \( [b_k, b_k^\dagger] = \delta_{kk'} \)
Hence we have an infinite # of linear coupled diffy-â's. We can formally integrate

\[ \hat{b}_k(t) = \hat{b}_k(0) e^{-i\omega_k t} - i q \int_0^t dt' \hat{a}(t') e^{-i\omega_k (t-t')} \]

Free evolution of Reservoir

Evolution of Reservoir perturbed by cavity.

plugging this back into \( \hat{a} \) gives

\[ \hat{a}(t) = -i \omega \hat{a}(t) - q^2 \sum_{k=1}^\infty \int_0^t dt' \hat{a}(t') e^{-i\omega_k (t-t')} + f(t) \]

Free evolution of cavity

"Driven" evolution by Reservoir

Noise

\[ \hat{f}(t) = -i q \sum_{k=1}^\infty \hat{b}_k(0) e^{-i \omega_k t} \]

The effect of the "driven" term is to cause a frequency shift (Lamb shift) but the noise \( \hat{f} \) causes loss (spontaneous emission).

We make the transformation

\[ \hat{a}(t) = \hat{a}(t) e^{i\omega_k t} \]

slowly varying envelope approximation

\[ \hat{a}(t) \]

\[ a(t) \]
\[ \begin{align*}
[ \hat{a}(t), \hat{a}^+(t) ] &= [ \hat{\tilde{a}}(t), \hat{\tilde{a}}^+(t) ] = 1 \quad \forall \ t \\
\Rightarrow \quad \hat{\dot{a}} &= -g^2 \sum_k \int_0^t dt' \hat{a}(t') e^{-i(\omega_k - \omega_0)(t-t')} + F(t) \\
F(t) &= e^{i\omega t} f(t) = -i g \sum_k b_k(t) e^{-i(\omega_k - \omega_0) t}
\end{align*} \]

We can again use the Fermi–Golden rule

\[ g^2 \sum_k \int_0^t dt' \hat{a}(t') e^{-i(\omega_k - \omega_0)(t-t')} \propto \delta(\omega - \omega_0) \]

where \( \rho(\omega_0) = N \cdot \frac{\omega^2}{\pi^2 c^3} \) is the Density of Modes

\[ \Rightarrow \quad \hat{\dot{a}}(t) = -\frac{1}{2} \gamma \hat{a}(t) + F(t) \]

\[ \gamma = 2\pi g^2 \rho(\omega_0) \]

Corresponds to Einstein A / photon emission

If \( F(t) \) is neg.

\[ \hat{\tilde{a}}(t) = \hat{\tilde{a}}(0) e^{-\frac{1}{2} \gamma t} \]

The field leaks out of the cavity with exponential decay
If reservoir at Temp $T = 0$

then fluctuation $\hat{F}$ correspond to vacuum fluctuation analogous to Landshoff shift

If $T > 0$ more complicated shift

$\Delta E < T^2$ the heat bath