6.5 Coherent state Interferometer

A Michelson Interferometer (MI) is mathematically $SU(2)$ equivalent to a Mach-Zehnder Interferometer (MZI)

Hence an MZI is called an "unfolded" MI which is easier to analyze. For $\phi = 0$, the interferometer is balanced so $\phi = \frac{\pi}{2} + k \cdot d$ where $d$ is the gravity wave displacement between two arms. Initially $\phi = 0$ and $p_3 = \cos^2(\frac{\pi}{c}) = 0$ and $p_2 = \sin^2(\frac{\pi}{c}) = 1$ so now 2 is the bright port and 3 the dark. For any $\phi < \pi$ the balance is destroyed and $p_3 = \cos^2(\frac{\pi}{c} + k \cdot d)$ and $p_2 = \sin^2(\frac{\pi}{c} + k \cdot d)$ will give us information about $d$. So measuring $\phi$ is equivalent to measuring $d$ our target.

Let us now send $|0\rangle_1 |\rangle_2 = |11\rangle$ as our input state. Recall that an MZI behaves as a BS with $\phi$ dependent $t$ and $r$ we recall:

$|\text{out}\rangle = |r\rangle_2 |t\rangle_3$ for a BS but for MZI $r = e^{i\phi/2} \cos(\phi/2)$ $t = e^{i\phi/2} \sin(\phi/2)$

Ignoring an overall phase $e^{i\phi}$
Let us again define \( \hat{n}_2 \) and \( \hat{n}_3 \) the detector operators.

\[
\langle \text{out} | \hat{a}_2 | \text{out} \rangle = \hat{x}_2 = \bar{n} \cos^2 \phi/2 \\
\langle \text{out} | \hat{a}_3 | \text{out} \rangle = \hat{x}_3 = \bar{n} \sin^2 \phi/2
\]

which exactly mimics the 1-photon case.

Note \( \hat{x}_2 + \hat{x}_3 = \bar{n} \) and energy is conserved.

Hence if \( \phi = \pi/2 \) all \( 1\rangle \) comes out \( \texttt{d} \) and none out \( \texttt{c} \).

A standard simple measurement scheme is to measure \( \hat{d} = \hat{n}_2 - \hat{n}_3 \) analogous to "inversion" with just the intensity difference between detectors.

\[
\langle \hat{d} \rangle = \bar{n} \cos^2 \phi/2 - \bar{n} \sin^2 \phi/2 = \bar{n} \cos \phi = f_\bar{n}(\phi)
\]

We would now like to estimate the minimum detectable phase \( \Delta \phi \). Assume that \( \langle \hat{d} \rangle = f_\bar{n}(\phi) \) is strongly peaked at \( \phi = \pi/2 \) and we can approximate \( \Delta \phi \approx \Delta f / \Delta \phi \) by differentials.

\[
\Delta \phi \approx \frac{\Delta f}{|df/d\phi|}
\]

which is what we get from error propagation theory.

Let us interpret \( \Delta f = \Delta d = \sqrt{\langle \hat{d}^2 \rangle - \langle \hat{d} \rangle^2} \) the variance of the quantum operator \( \hat{d} \).
\[ \langle \hat{a}^2 \rangle = \langle \text{out} | (\hat{n}_2 - \hat{n}_3)^2 | \text{out} \rangle \]

\[ = \langle \text{out} | \hat{n}_2^2 - 2 \hat{n}_2 \hat{n}_3 + \hat{n}_3^2 | \text{out} \rangle \]

\[ = \langle \text{out} | \left[ \hat{a}_2^+ \hat{a}_2 \hat{a}_3^+ \hat{a}_3 - 2 \hat{n}_2 \hat{n}_3 + \hat{a}_3^+ \hat{a}_3 \hat{a}_2^+ \hat{a}_2 \right] | \text{out} \rangle \]

\[ = \langle \text{out} | \left[ \hat{a}_2^+ (\hat{a}_2 \hat{a}_3 + 1) \hat{a}_3 - 2 \hat{n}_2 \hat{n}_3 + \hat{a}_3^+ \hat{a}_3 \hat{a}_2^+ \hat{a}_2 \right] | \text{out} \rangle \]

\[ = \langle \text{out} | \left[ \hat{a}_2^+ \hat{a}_2 + \hat{a}_3^+ \hat{a}_3 - 2 \hat{n}_2 \hat{n}_3 + \hat{a}_3^+ \hat{a}_3 \hat{a}_2^+ \hat{a}_2 \right] | \text{out} \rangle \]

\[ = \langle \hat{d} \rangle^2 + \overline{n} \]

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\[ \overline{n} = \text{Vacuum term} \]

\[ \Delta d = \langle \hat{d} \rangle - \langle \hat{d} \rangle^2 = \overline{n} = \text{Vacuum term} \]

\[ \Delta \phi = \frac{\Delta \hat{d}}{\partial \langle \hat{d} \rangle / \partial \phi} \]

\[ \Delta \phi = \frac{1}{\sqrt{\overline{n}}} \]

We have already seen \( \phi \propto \pi / 2 \) is our working point for a balanced MOSFET so

\[ \Delta \phi = \frac{1}{\sqrt{\overline{n}}} \]

\[ \Delta l = \frac{\lambda}{\sqrt{\overline{n}}} \]

\[ \overline{n} = I = \text{circuiting power} \approx 1 \text{ kW! But vacuum determines sensitivity!} \]