"I think you should be more explicit here in step two."
(1) [20pts] Consider a two-dimensional vector space spanned by an orthonormal basis \( |1\rangle \) and \( |2\rangle \). We define two kets \( |\alpha\rangle = |1\rangle - i|2\rangle \) and \( |\beta\rangle = i|1\rangle + 2|2\rangle \).

(a) [5pts] Construct \( \langle \alpha | \) and \( \langle \beta | \) in terms of the dual basis \( \langle 1 | \) and \( \langle 2 | \).

\[
\langle \alpha | = -\frac{i}{2} |1\rangle + \frac{1}{2} |2\rangle \\
\langle \beta | = -\frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle
\]

(b) [5pts] Find \( \langle \alpha | \beta \rangle \) and \( \langle \beta | \alpha \rangle \), and confirm that \( \langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^* \).

\[
\langle \alpha | \beta \rangle = \left[ -\frac{i}{2} |1\rangle + \frac{1}{2} |2\rangle \right] \left[ \frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle \right] = \frac{1}{4} (1 + i) + \frac{1}{4} (1 - i) = \frac{1}{2} \\
\langle \beta | \alpha \rangle = \left[ -\frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle \right] \left[ \frac{1}{2} |1\rangle + \frac{i}{2} |2\rangle \right] = \frac{1}{4} (1 - i) + \frac{1}{4} (1 + i) = \frac{1}{2}
\]

\( \langle \beta | \alpha \rangle^* = \langle \alpha | \beta \rangle \checkmark \)

(c) [5pts] Find all four matrix elements of the operator \( \hat{\mathcal{A}} \), in this basis, and construct the matrix \( \mathcal{A} \) and the matrix \( \mathcal{A}^\dagger \). Is the matrix \( \mathcal{A} \) Hermitian? Why or why not?

\[
\begin{align*}
\mathcal{A}_{11} &= \langle 1 | \hat{\mathcal{A}} | 1 \rangle = \langle 1 | 1\rangle = 1 \\
\mathcal{A}_{12} &= \langle 1 | \hat{\mathcal{A}} | 2 \rangle = \langle 1 | i \rangle = 0 \\
\mathcal{A}_{21} &= \langle 2 | \hat{\mathcal{A}} | 1 \rangle = \langle 2 | 1\rangle = 0 \\
\mathcal{A}_{22} &= \langle 2 | \hat{\mathcal{A}} | 2 \rangle = \langle 2 | 2\rangle = 1
\end{align*}
\]

\[
\mathcal{A} = \begin{bmatrix}
1 & i \\
-1 & -2i
\end{bmatrix} \quad \mathcal{A}^\dagger = \begin{bmatrix}
1 & -i \\
-1 & 2i
\end{bmatrix} \neq \mathcal{A}
\]

\( \mathcal{A} \text{ is not Hermitian} \)

(d) [5pts] Find the two eigenvalues of the matrix \( \mathcal{A} \). Are the eigenvalues real or complex? Is your result consistent with the result of (c)? Why or why not?

\[
\det \begin{vmatrix}
1 - \lambda & 2i \\
-1 & -2i - \lambda
\end{vmatrix} = 0 \Rightarrow (\lambda + 2i)(\lambda - 1) + 2i = 0
\]

\[
\Rightarrow \lambda^2 + (2i - 1)\lambda - 2i + 2i = 0 \\
\Rightarrow \lambda \left[ \lambda + (2i - 1) \right] = 0
\]

\[
\begin{align*}
\lambda &= 0 \\
\lambda_2 &= i - \frac{1}{2}
\end{align*}
\]

\( \lambda_2 \text{ not real} \Rightarrow \mathcal{A} \text{ not Hermitian} \)

\( \lambda_2 \text{ not real} \Rightarrow \mathcal{A} \text{ not Hermitian} \)
(2) [20 pts] Use separation of variables in Cartesian coordinates in 2D to solve the infinite square well (or "particle in a box"): \(V(x,y) = 0\) if \(0 \leq x, y \leq a\) and infinity on and outside the square.

(a) [5 pts] Write down the 2-D TISE for inside the box.

\[
-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = E\psi
\]

Let \(\mathbf{\hat{\mathbf{r}}}^2 = 2m\hbar^2 E\).

(b) [5 pts] Use the separation ansatz \(\psi(x, y) = X(x)Y(y)\) and derive two separate ordinary differential equations for \(X\) and \(Y\) in terms of \(k_x\) and \(k_y\) where \(E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2)\).

\[
\begin{align*}
\Psi(x, y) &= X(x)Y(y) \quad \text{satisfy} \quad -\frac{\hbar^2}{2m} \left[ \frac{d^2}{dx^2}X + \frac{d^2}{dy^2}Y \right] = E X Y \\
\Rightarrow \quad X'' + X'' &= -k_x^2 X \quad \Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = -k_x^2 \\
\Rightarrow \quad \frac{X''}{X} = -k_x^2 \quad \frac{Y''}{Y} = -k_y^2 \quad \text{constr. sep.} \\
\Rightarrow \quad X'' + X_x X &= 0 \quad Y'' + k_y Y = 0 \\
\text{By}\quad k_x^2 + k_y^2 &= \frac{E}{\hbar^2} = \frac{2ny^2 \hbar^2}{E}
\end{align*}
\]

(c) [5 pts] From the boundary conditions derive an expression for the quantized eigenenergies \(E\) of the states in terms of the quantum numbers \(n_x, n_y = 1, 2, 3, \ldots\). Hint: you also can get this by demanding a half integer number of wavelengths must fit across each direction in the box.

\[
\begin{align*}
X &= A_x e^{i k_x x} \cos k_x x \quad \text{at} \quad x = 0 \quad \Rightarrow \quad B_x = 0, \quad k_x a = \pi n_x \\
Y &= A_y e^{i k_y y} \cos k_y y \quad \text{at} \quad y = 0 \quad \Rightarrow \quad B_y = 0, \quad k_y a = \pi n_y \\
\Rightarrow \quad n_x, n_y &= 1, 2, \ldots \\
\text{Thus} \quad E &= \frac{\hbar^2}{2m} \left[ \frac{\pi^2 n_x^2}{a^2} + \frac{\pi^2 n_y^2}{a^2} \right]
\end{align*}
\]

(d) [5 pts] Apply the boundary conditions at the walls of the box and solve for in terms of \(x, y\), and the quantum numbers \(n_x, n_y\). Are these degenerate or non-degenerate? Explain.

\[
\Psi_{n_x n_y}(x, y) = A_x A_y \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{a} \right)
\]

\[
\begin{align*}
\Psi_{11} &= \Psi_{21} \\
\text{But} \quad E_{11} &= E_{21}
\end{align*}
\]
(3) [20 pts] The ground state of hydrogen is 
\[ \psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \text{ where } a \text{ is the Bohr radius.} \]

\[ |\psi|^2 = e^{-r/a} \frac{1}{(\pi a^3)} \]

(a) [5 pts] Plot the probability density \( |\psi|^2 \) as a function of \( r/a \) on the graph. Indicate the value of the curve at \( r = 0 \). Where is the maximum of this curve?

(b) [5 pts] Find the most likely position \( \langle r \rangle \) for this state. Does this correspond to the maximum of the curve in (a)? Why or why not. Hint: \[ \frac{dV}{dr} = 3 r^{-3} e^{-2r/a} \] and \[ \int_{0}^{a} r^{-3} e^{-2r/a} dr = \frac{3}{10} \frac{a}{r^2} \]

\[ \langle r \rangle = \int r |\psi|^2 dV = \int \frac{1}{\pi a^3} r \frac{3}{10} e^{-r/a} \frac{a}{r^2} \]

\[ = \frac{3 a^4}{10} \]

\[ \text{Prob is } |\psi|^2 dV \sim |\psi|^2 r^2 = \rho(r) \]

\[ \rho(r) \sim |\psi|^2 dV \approx r^2 \]

(c) [5 pts] Find \( \langle r^2 \rangle \) and the uncertainty in position \( \Delta r \) for this state. What can you say about the uncertainty in momentum \( \Delta p \)? Hint: \[ \frac{dH_r}{dr} = \frac{3 a}{r^2} \]

\[ \langle r^2 \rangle = \int r^2 |\psi|^2 dV = \frac{3 a^4}{10} \]

\[ \Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} = \sqrt{\frac{3 a^2}{4}} - \frac{a^2}{4} = \frac{a^2}{4} \]

\[ \Delta r = \frac{a^2}{\sqrt{13}} \]

(d) [5 pts] Find \( \langle x \rangle \), \( \langle x^2 \rangle \), and \( \Delta x \) for this state. What can you say about \( \Delta p_x \)? Hint: no new integration. Note that \( r^2 = x^2 + y^2 + z^2 \), and exploit the spherical symmetry of this state.

\[ \langle x \rangle = \langle y \rangle = \langle z \rangle = 0 \text{ sym.} \]

\[ \langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle \]

\[ \Rightarrow \langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = \frac{a^2}{3} \]

\[ \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a \]

\[ \frac{1}{2} \Delta p_x = \frac{\hbar}{2 a} \]

\[ \Rightarrow \Delta p_x \geq \frac{\hbar}{2 a} \]
(4) [20pts] Two identical particles named Alice and Bob, both in the same spin state, \( |\uparrow\rangle, |\downarrow\rangle \), are put into the 1D infinite square well as shown. The one-particle ground states for each particle are 
\[
\psi_a(x_1) = \sqrt{2/a} \sin(\pi x_1/a) \quad \text{and} \quad \psi_b(x_2) = \sqrt{2/a} \sin(\pi x_2/a),
\]
each with energy 
\[
E = \frac{\pi^2 \hbar^2}{2ma^2}.
\]

(a) [5pts] Write down the correct two-particle ground state spatial wavefunction \( \psi_{ab}(x_1, x_2) \) if the two particles are distinguishable (classical). What is the energy of this two-particle state in terms of \( E \)?
\[
\psi_{AB}(x_1, x_2) = \psi_a(x_1) \psi_b(x_2) = \left[ \frac{2}{a} \sin(\pi x_1/a), \frac{2}{a} \sin(\pi x_2/a) \right] \quad \text{as \ } E = K + U.
\]

(b) [5pts] Write down the correct two-particle ground state spatial wavefunction \( \psi_{ab}(x_1, x_2) \) if the two particles are indistinguishable bosons with spin one. Would you expect them to be on average closer together or further apart than in (a)? Why?
\[
\psi_{AB}(x_1, x_2) = \psi_a(x_1) \psi_b(x_2) \otimes \psi_a(x_2) \psi_b(x_1) = \left[ \frac{2}{a} \sin(\pi x_1/a), \frac{2}{a} \sin(\pi x_2/a) \right] \quad \text{SAME AS CLASSICAL}
\]

(c) [5pts] Write down the correct two-particle ground state spatial wavefunction \( \psi_{ab}(x_1, x_2) \) if the two particles are indistinguishable fermions with spin one-half. Is your answer consistent with the Pauli exclusion principle? Why or why not?
\[
\psi_{AB}(x_1, x_2) = \psi_a(x_1) \psi_b(x_2) \otimes \psi_a(x_2) \psi_b(x_1) = \left[ \frac{2}{a} \sin(\pi x_1/a), \frac{2}{a} \sin(\pi x_2/a) \right] \quad \text{SUCH A STATE CANNOT EXIST!}
\]

(d) [5pts] Write down the correct two-particle ground state spatial wavefunction \( \psi_{ab}(x_1, x_2) \) if the two particles are indistinguishable fermions with spin one-half prepared in the spin-singlet state \( \left( |\uparrow\rangle_a |\downarrow\rangle_b - |\downarrow\rangle_a |\uparrow\rangle_b \right)/\sqrt{2} \). Would you expect them to be on average closer together or further apart than in (a)? Why?
\[
\psi_{AB}(x_1, x_2) = \frac{2}{a} \sin(\pi x_1/a), \frac{2}{a} \sin(\pi x_2/a) \quad \text{SINCE SPIN STATE ANTI SYMMETRIC NOW SPATIAL STATE MUST BE SYMMETRIC.}
\]
Problem 1 [20 points] The Planck blackbody spectrum for the photon energy density $N_{ph} \nu^3 \frac{d\nu}{E}$ in the frequency range $d\nu$ is

$$\rho(\nu) d\nu = \frac{h\nu^3}{\pi^2 c^3} \frac{\nu}{e^{\frac{h\nu}{kT}} - 1} \cdot \frac{3}{\pi^2 c^3} d\nu$$

(b) [5 points] In the figure are sketches of $\rho(\nu)$ for $T = 2000K^\circ$, $4000K^\circ$, and $6000K^\circ$. Label which curve is at which temperature.

(b) [5 points] Solve $\rho'(\nu) = 0$ and find an approximate expression for $\nu_{\text{max}}$, the frequency where the curve is maximum, as a function of $h$, $c$, $k_B$, and $T$. Is your result consistent with the plots in (a)? Why or why not?

$$0 = \rho'(\nu) = \left[ \frac{3}{\pi^2 c^3} \frac{(e^{\nu/kT} - 1)^2 - 3e^{\nu/kT}e^{\nu/kT}}{(e^{\nu/kT} - 1)^2} \right] = 0$$

$$\Rightarrow \frac{3}{\pi^2 c^3} \left[ \frac{(e^{\nu/kT} - 1)^2 - 3e^{\nu/kT}e^{\nu/kT}}{(e^{\nu/kT} - 1)^2} \right] = 0$$

$$\Rightarrow \frac{3}{\pi^2 c^3} \left[ \frac{(e^{\nu/kT} - 1)^2 - 3(e^{\nu/kT} - 1) - 3(e^{\nu/kT} - 1)^2}{(e^{\nu/kT} - 1)^2} \right] = 0$$

$$\Rightarrow \frac{3}{\pi^2 c^3} \left[ \frac{(e^{\nu/kT} - 1)^2 - 3(e^{\nu/kT} - 1) - 3(e^{\nu/kT} - 1)^2}{(e^{\nu/kT} - 1)^2} \right] = 0$$

(c) [5 points] Using unit analysis only give an approximate formula in terms of $h$, $c$, $k_B$, and $T$, for the wavelength $\lambda_{\text{max}}$ where the energy density is a maximum. (Hint: $h$ has units of $J$ and $k_B$, and $T$.) Use this to give an approximate expression for the frequency $\nu_{\text{max}}$ where the energy density is a maximum. Is the result consistent with your plots in (b)? Why or why not?

$$\lambda_{\text{max}} = \left[ \frac{m}{\lambda} \right] = \left[ \frac{m}{\lambda} \right] = \left[ \frac{c}{h \nu} \right]$$

$$\omega_{\text{max}} = \frac{\lambda_{\text{max}}}{c} \Rightarrow \omega_{\text{max}} = \frac{c \nu}{\lambda_{\text{max}}} \Rightarrow \lambda_{\text{max}} = \frac{h \nu}{c} \Rightarrow \lambda_{\text{max}} = \frac{h \nu}{c}$$

(d) [5 points] Derive an exact formula for the total energy density $E/V$ in terms of $h$, $c$, $k_B$, and $T$. (Hint: $\int_0^\infty e^{\nu/kT} d\nu = \pi^2 / 15$.) Is your result consistent with (a)? Why or why not?

$$E = \int_0^\infty \rho(\nu) d\nu = \frac{(h \nu)^3}{(k_B T)^2} \int_0^\infty \frac{\nu^3}{e^{\nu/kT} - 1} d\nu = \frac{(h \nu)^4}{(k_B T)^2} \left[ \frac{\nu^4}{4} \right]_0^\infty = \frac{(h \nu)^4}{(k_B T)^2} \left[ \frac{\nu^4}{15} \right]_0^\infty$$

$$\Rightarrow$$

$$\Rightarrow$$

Yes area under curve inc. with inc. $T$.